AFFINE HECKE ALGEBRAS FOR LANGLANDS PARAMETERS

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Abstract. It is well-known that affine Hecke algebras are very useful to describe the smooth representations of any connected reductive p-adic group $\mathcal{G}$, in terms of the supercuspidal representations of its Levi subgroups. The goal of this paper is to create a similar role for affine Hecke algebras on the Galois side of the local Langlands correspondence.

To every Bernstein component of enhanced Langlands parameters for $\mathcal{G}$ we canonically associate an affine Hecke algebra (possibly extended with a finite R-group). We prove that the irreducible representations of this algebra are naturally in bijection with the members of the Bernstein component, and that the set of central characters of the algebra is naturally in bijection with the collection of cuspidal supports of these enhanced Langlands parameters. These bijections send tempered or (essentially) square-integrable representations to the expected kind of Langlands parameters.

Furthermore we check that for many reductive p-adic groups, if a Bernstein component $\mathfrak{B}$ for $\mathcal{G}$ corresponds to a Bernstein component $\mathfrak{B}^\vee$ of enhanced Langlands parameters via the local Langlands correspondence, then the affine Hecke algebra that we associate to $\mathfrak{B}^\vee$ is Morita equivalent with the Hecke algebra associated to $\mathfrak{B}$. This constitutes a generalization of Lusztig's work on unipotent representations. It might be useful to establish a local Langlands correspondence for more classes of irreducible smooth representations.

Contents

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Introduction

Let F be a local non-archimedean field and let $\mathcal{G}$ be a connected reductive algebraic group defined over F. The conjectural local Langlands correspondence (LLC) provides a bijection between the set of irreducible smooth $\mathcal{G}(F)$-representations $\mathrm{Irr}(\mathcal{G}(F))$ and the set of enhanced L-parameters $\Phi_e(\mathcal{G}(F))$, see [Bor, Vog, ABPS5].

Let $\mathfrak{s}$ be an inertial equivalence class for $\mathcal{G}(F)$ and let $\mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}}$ be the associated Bernstein component. Similarly, inertial equivalence classes $\mathfrak{s}^\vee$ and Bernstein components $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$ for enhanced L-parameters were developed in [AMS1]. It can be expected that every $\mathfrak{s}$ corresponds to a unique $\mathfrak{s}^\vee$ (an "inertial Langlands correspondence"), such that the LLC restricts to a bijection

$$\mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}} \longleftrightarrow \Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}. \tag{1}$$

The left hand side can be identified with the space of irreducible representations of a direct summand $\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}$ of the full Hecke algebra of $\mathcal{G}(F)$. It is known that in many cases $\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}$ is Morita equivalent to an affine Hecke algebra, see [ABPS5, §2.4] and the references therein for an overview.

To improve our understanding of the LLC, we would like to canonically associate to $\mathfrak{s}^\vee$ an affine Hecke algebra $\mathcal{H}(\mathfrak{s}^\vee)$ whose irreducible representations are naturally parametrized by $\Phi_e(\mathcal{G}(F))$. Then (1) could be written as

$$\mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}} \cong \mathrm{Irr}\big(\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}\big) \longleftrightarrow \mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee)) \cong \Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}, \tag{2}$$

and the LLC for this Bernstein component would become a comparison between two algebras of the same kind. If moreover $\mathcal{H}(\mathfrak{s}^\vee)$ were Morita equivalent to $\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}$, then (1) could even be categorified to

$$\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}} \cong \mathrm{Mod}(\mathcal{H}(\mathfrak{s}^\vee)). \tag{3}$$

Such algebras $\mathcal{H}(\mathfrak{s}^\vee)$ would also be useful to establish the LLC in new cases. One could compare $\mathcal{H}(\mathfrak{s}^\vee)$ with the algebras $\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}$ for various $\mathfrak{s}$, and only the Bernstein components $\mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}}$ for which $\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}$ is Morita equivalent with $\mathcal{H}(\mathfrak{s}^\vee)$ would be good candidates for the image of $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$ under the LLC. If one would know a lot about $\mathcal{H}(\mathfrak{s}^\vee)$, this could substantially reduce the number of possibilities for a LLC for these parameters.

This strategy was already employed by Lusztig, for unipotent representations [Lus4, Lus5]. Bernstein components of enhanced L-parameters had not yet been defined when the papers [Lus4, Lus5] were written, but the constructions in them can be interpreted in that way. Lusztig found a bijection between:

- the set of ("arithmetic") affine Hecke algebras associated to unipotent Bernstein blocks of adjoint, unramified groups;
- the set of ("geometric") affine Hecke algebras associated to unramified enhanced L-parameters for such groups.

The comparison of Hecke algebras is not enough to specify a canonical bijection between Bernstein components on the p-adic and the Galois sides. The problem is that one affine Hecke algebra can appear (up to isomorphism) several times on either side. This already happens in the unipotent case for exceptional groups, and the issue seems to be outside the scope of these techniques. In [Lus4, 6.6–6.8] Lusztig wrote down some remarks about this problem, but he does not work it out completely.

The main goal of this paper is the construction of an affine Hecke algebra for any Bernstein component of enhanced L-parameters, for any $\mathcal{G}$. But it quickly turns out that this is not precisely the right kind of algebra. Firstly, our geometric construction, which relies on [Lus2, AMS2], naturally includes some complex parameters $\mathbf{z}_i$, which we abbreviate to $\vec{\mathbf{z}}$. Secondly, an affine Hecke algebra with (indeterminate) parameters is still too simple. In general one must consider the crossed product of such an object with a twisted group algebra (of some finite "R-group"). We call this a twisted affine Hecke algebra, see Proposition 2.2 for a precise definition. Like for reductive groups, there are good notions of tempered representations and of (essentially) discrete series representations of such algebras (Definition 2.6).

Theorem 1. *[see Theorem 3.15]*

(a) *To every Bernstein component of enhanced L-parameters $\mathfrak{s}^\vee$ one can canonically associate a twisted affine Hecke algebra $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$.*

(b) *For every choice of parameters $z_i \in \mathbb{R}_{>0}$ there exists a natural bijection*

$$\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee} \leftrightarrow \mathrm{Irr}\big(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})/(\{\mathbf{z}_i - z_i\}_i)\big)$$

(c) *For every choice of parameters $z_i \in \mathbb{R}_{\geq 1}$ the bijection from part (b) matches enhanced bounded L-parameters with tempered irreducible representations.*

(d) *For every choice of parameters $z_i \in \mathbb{R}_{>1}$ the bijection from part (b) matches enhanced discrete L-parameters with irreducible essentially discrete series representations.*

This can be regarded as a far-reaching generalization of parts of [Lus4, Lus5]: we allow any reductive group over a local non-archimedean field, and all enhanced L-parameters for that group. We check (see Section 5) that in several cases where the LLC is known, indeed

$$\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}} \quad \text{is Morita equivalent to} \quad \mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})/(\{\mathbf{z}_i - z_i\}_i)\big) \tag{4}$$

for suitable $z_i \in \mathbb{R}_{>1}$, obtaining (3). Notice that on the p-adic side the parameters z_i are determined by $\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}$, whereas on the Galois side we specify them manually.

Yet in general the categorification (3) is asking for too much. We discovered that for inner twists of $\mathrm{SL}_n(F)$ (4) does not always hold. Rather, these algebras are equivalent in a weaker sense: the category of finite length modules of $\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}$ (i.e. the finite length objects in $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$) is equivalent to the category of finite dimensional representations of $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})/(\{\mathbf{z}_i - z_i\}_i)$, again for suitable $z_i \in \mathbb{R}_{>1}$.

Let us describe the contents of the paper more concretely. Our starting point is a triple $(G, M, q\mathcal{E})$ where

- G is a possibly disconnected complex reductive group,
- M is a quasi-Levi subgroup of G (the appropriate possibly disconnected analogue of a Levi subgroup),
- $q\mathcal{E}$ is a M-equivariant cuspidal local system on a unipotent orbit $\mathcal{C}_v^M$ in M.

To these data we attach a twisted affine Hecke algebra $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$. This algebra can be specialized by setting $\vec{\mathbf{z}}$ equal to some $\vec{z} \in (\mathbb{C}^\times)^d$. Of particular interest is the specialization at $\vec{z} = \vec{1}$:

$$\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})/(\{\mathbf{z}_i - 1\}_i) = \mathcal{O}(T) \rtimes \mathbb{C}[W_{q\mathcal{E}}, \natural],$$

where $T = Z(M)^\circ$, while the subgroup $W_{q\mathcal{E}} \subset N_G(M)/M$ and the 2-cocycle $\natural : W_{q\mathcal{E}}^2 \to \mathbb{C}^\times$ also come from the data.

The goal of Section 2 is to understand and parametrize representations of $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$. We follow a strategy similar to that in [Lus3]. The centre naturally contains $\mathcal{O}(T)^{W_{q\mathcal{E}}} = \mathcal{O}(T/W_{q\mathcal{E}})$, so we can study $\mathrm{Mod}(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}))$ via localization at suitable subsets of $T/W_{q\mathcal{E}}$. In Paragraph 2.1 we reduce to representations with $\mathcal{O}(T)^{W_{q\mathcal{E}}}$-character in $W_{q\mathcal{E}} T_{\mathrm{rs}}$, where T_{rs} denotes the maximal real split subtorus of T. This involves replacing $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ by an algebra of the same kind, but for a smaller G.

In Paragraph 2.2 we reduce further, to representations of a (twisted) graded Hecke algebra $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$. We defined and studied such algebras in our previous paper [AMS2]. But there we only considered the case with a single parameter $\mathbf{r}$, here we need $\vec{\mathbf{r}} = (\mathbf{r}_1, \ldots, \mathbf{r}_d)$. The generalization of the results of [AMS2] to a multi-parameter setting is carried out in Section 1. With that at hand we can use the construction of "standard" $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$-modules and the classification of irreducible $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$-modules from [AMS2] to achieve the same for $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$. For the parametrization we use triples (s, u, ρ) where:

- $s \in G^\circ$ is semisimple,
- $u \in Z_G(s)^\circ$ is unipotent,
- $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(s, u))\big)$ such that the quasi-cuspidal support of (u, ρ), as defined in [AMS1, §5], is G-conjugate to $(M, \mathcal{C}_v^M, q\mathcal{E})$.

Theorem 2. *[see Theorem 2.11]*

(*a*) *Let $\vec{z} \in \mathbb{R}^d_{\geq 0}$. There exists a canonical bijection, say $(s, u, \rho) \mapsto \bar{M}_{s,u,\rho,\vec{z}}$, between:*
  - *G-conjugacy classes of triples (s, u, ρ) as above,*
  - $\mathrm{Irr}\big(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})/(\{\mathbf{z}_i - z_i\}_i)\big)$.

(*b*) *Let $\vec{z} \in \mathbb{R}^d_{\geq 1}$. The module $\bar{M}_{s,u,\rho,\vec{z}}$ is tempered if and only if s is contained in a compact subgroup of G°.*

(*c*) *Let $\vec{z} \in \mathbb{R}^d_{> 1}$. The module $\bar{M}_{s,u,\rho,\vec{z}}$ is essentially discrete series if and only if u is distinguished unipotent in G° (i.e. does not lie in a proper Levi subgroup).*

In the case $M = T$, $\mathcal{C}_v^M = \{1\}$ and $q\mathcal{E}$ trivial, the irreducible representations in $\mathcal{H}(G^\circ, T, q\mathcal{E} = \mathrm{triv})$ were already classified in the landmark paper [KaLu], in terms of similar triples. In Paragraph 2.3 we check that the parametrization from Theorem 2 agrees with the Kazhdan–Lusztig parametrization for these algebras.

Remarkably, our analysis also reveals that [KaLu] does not agree with the classification of irreducible representations in [Lus4]. To be precise, the difference consists of a twist with a version of the Iwahori–Matsumoto involution. Since [KaLu] is widely regarded (see for example [Ree, Vog]) as the correct local Langlands correspondence for Iwahori-spherical representations, this entails that the parametrizations obtained by Lusztig in [Lus4, Lus5] can be improved by composition with a suitable involution. In the special case $G = \mathrm{Sp}_{2n}(\mathbb{C})$, that already transpired from work of Mœglin and Waldspurger [Wal].

Having obtained a good understanding of affine Hecke algebras attached to disconnected reductive groups, we turn to Langlands parameters. Let

$$\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathcal{G}$$

be a L-parameter and let ρ be an enhancement of ϕ. (See Section 3 for the precise notions.) Let $\mathcal{G}_{\mathrm{sc}}$ be the simply connected cover of the derived group of the complex dual group $\mathcal{G}^\vee$ and consider $G = Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi(\mathbf{I}_F))$. We emphasize that this complex group can be disconnected - this is the main reason why we have to investigate Hecke algebras for disconnected reductive groups.

Recall that ϕ is determined up to $\mathcal{G}^\vee$-conjugacy by $\phi|_{\mathbf{W}_F}$ and the unipotent element $u_\phi = \phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big)$. As the image of a Frobenius element is allowed to vary within one Bernstein component, $(\phi|_{\mathbf{I}_F}, u_\phi)$ contains almost all information about such a Bernstein component.

The cuspidal support of (u_ϕ, ρ) for G is a triple $(M, \mathcal{C}_v^M, q\mathcal{E})$ as before. Thus we can associate to (ϕ, ρ) the twisted affine Hecke algebra $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$. This works quite well in several cases, but in general it is too simple, we encounter various technical difficulties. The main problem is that the torus $T = Z(M)^\circ$ will not always match up with the torus from which the Bernstein component of $\Phi_e(\mathcal{G}(F))$ containing (ϕ, ρ) is built.

Instead we consider the twisted graded Hecke algebra $\mathbb{H}(Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi(\mathbf{W}_F)), M, q\mathcal{E}, \vec{\mathbf{r}})$, and we tensor it with the coordinate ring of a suitable vector space to compensate for the difference between $\mathcal{G}^\vee_{\mathrm{sc}}$ and $\mathcal{G}^\vee$. In Paragraph 3.1 we prove that the irreducible representations of the ensuing algebra are naturally parametrized by a subset of the Bernstein component $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$ containing (ϕ, ρ). In Paragraph 3.2 we glue a families of such algebras together, to obtain the twisted affine Hecke algebras $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$ featuring in Theorem 1.

Let us compare our paper with similar work by other authors. Several mathematicians have noted that, when two Bernstein components give rise to isomorphic affine Hecke algebras, this often has to do with the centralizers of the corresponding Langlands parameters. It is known from the work of Bushnell–Kutzko (see in particular [BuKu2]) that every affine Hecke algebra associated to a semisimple type for $\mathrm{GL}_n(F)$ is isomorphic to the Iwahori–spherical Hecke algebra of some $\prod_i \mathrm{GL}_{n_i}(F_i)$, where $\sum_i n_i \leq n$ and F_i is a finite extension of the field F. A similar statement holds for Bernstein components in the principal series of F-split reductive groups [Roc, Lemma 9.3].

Dat [Dat, Corollary 1.1.4] has generalized this to groups of "GL-type", and in [Dat, Theorem 1.1.2] he proves that for such a group $Z_{\mathcal{G}^\vee}(\phi(\mathbf{I}_F))$ determines $\prod_{\mathfrak{s}} \mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$, where $\mathfrak{s}$ runs over all Bernstein components that correspond to extensions of $\phi|_{\mathbf{I}_F}$ to $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$. In [Dat, §1.3] Dat discusses possible generalizations of these results to other reductive groups, but he did not fully handle the cases where $Z_{\mathcal{G}^\vee}(\phi(\mathbf{I}_F))$ is disconnected. (It is always connected for groups of GL-type.) Theorem 1, in combination with the considerations about inner twists of $\mathrm{GL}_n(F)$ in Paragraph 5.1, provide explanations for all the equivalences between Hecke algebras and between categories found by Dat.

Heiermann [Hei2, §1] has associated affine Hecke algebras (possibly extended with a finite R-group) to certain collections of enhanced L-parameters for classical groups (essentially these sets constitute unions of Bernstein components). Unlike Lusztig he does not base this on geometric constructions in complex groups, rather on affine Hecke algebras previously found on the p-adic side in [Hei1]. In his setup (2) holds true by construction, but the Hecke algebras are only related to L-parameters via the LLC, so not in an explicit way.

In [Hei2, §2] it is shown that every Bernstein component of enhanced L-parameters for a classical group is in bijection with the set of unramified L-parameters for $Z_{\mathcal{G}^\vee}(\phi(\mathbf{I}_F))$, which is the complex dual of a product of classical groups of smaller rank. So in the context of [Hei2] the data that we use for affine Hecke algebras are present, and the algebras are also present (at least up to Morita equivalence), but the link between them is not yet explicit. In Paragraph 5.3 we discuss how our results clarify this.

1. Twisted graded Hecke algebras

We will recall some aspects of the (twisted) graded Hecke algebras studied in [AMS2]. Let G be a complex reductive group, possibly disconnected. Let M be a quasi-Levi subgroup of G, that is, a group of the form $M = Z_G(Z(L)^\circ)$ where L is a Levi subgroup of G°. Notice that $M^\circ = L$ in this case.

We write $T = Z(M)^\circ = Z(M^\circ)^\circ$, a torus in G°. Let $P^\circ = M^\circ U$ be a parabolic subgroup of G° with Levi factor M° and unipotent radical U. We put $P = MU$. Let $\mathfrak{t}^*$ be the dual space of the Lie algebra $\mathfrak{t} = \mathrm{Lie}(T)$.

Let $v \in \mathfrak{m} = \mathrm{Lie}(M)$ be nilpotent, and denote its adjoint orbit by $\mathcal{C}_v^M$. Let $q\mathcal{E}$ be an irreducible M-equivariant cuspidal local system on $\mathcal{C}_v^M$. Then the stalk $q\epsilon = q\mathcal{E}|_v$ is an irreducible representation of $A_M(v) = \pi_0(Z_M(v))$. Conversely, v and $q\epsilon$ determine $\mathcal{C}_v^M$ and $q\mathcal{E}$. By definition the cuspidality means that $\mathrm{Res}^{A_M(v)}_{A_{M^\circ}(v)} q\epsilon$ is a direct sum of irreducible cuspidal $A_{M^\circ}(v)$-representations. Let $\epsilon \in \mathrm{Irr}(A_{M^\circ}(v))$ be one of them, and let $\mathcal{E}$ be the corresponding M°-equivariant cuspidal local system on $\mathcal{C}_v^{M^\circ}$. Then $\mathcal{E}$ is a subsheaf of $q\mathcal{E}$. See [AMS1, §5] for more background.

The triple $(M, \mathcal{C}_v^M, q\mathcal{E})$ (or $(M, v, q\epsilon)$) is called a cuspidal quasi-support for G. We denote its G-conjugacy class by $[M, \mathcal{C}_v^M, q\mathcal{E}]_G$. With these data we associate the groups

$$\begin{aligned} N_G(q\mathcal{E}) &= \mathrm{Stab}_{N_G(M)}(q\mathcal{E}), \\ W_{q\mathcal{E}} &= N_G(q\mathcal{E})/M, \\ W_{q\mathcal{E}}^\circ &= N_{G^\circ M}(M)/M \cong N_{G^\circ}(M^\circ)/M^\circ = W_{\mathcal{E}}^\circ, \\ \mathfrak{R}_{q\mathcal{E}} &= \big(N_G(q\mathcal{E}) \cap N_G(P, M)\big)/M. \end{aligned} \tag{5}$$

The group $W_{q\mathcal{E}}$ acts naturally on the set

$$R(G^\circ, T) := \{\alpha \in X^*(T) \setminus \{0\} : \alpha \text{ appears in the adjoint action of } T \text{ on } \mathfrak{g}\}.$$

By [Lus1, Theorem 9.2] (see also [AMS2, Lemma 2.1]) $R(G^\circ, T)$ is a root system with Weyl group $W_{q\mathcal{E}}^\circ$. The group $\mathfrak{R}_{q\mathcal{E}}$ is the stabilizer of the set of positive roots determined by P and

$$W_{q\mathcal{E}} = W_{q\mathcal{E}}^\circ \rtimes \mathfrak{R}_{q\mathcal{E}}.$$

We choose semisimple subgroups $G_j \subset G^\circ$, normalized by $N_G(q\mathcal{E})$, such that the derived group G°_{der} is the almost direct product of the G_j. In other words, every G_j is semisimple, normal in $G^\circ M$, normalized by $W_{q\mathcal{E}}$ (which makes sense because it is already normalized by M), and the multiplication map

$$m_{G^\circ} : Z(G^\circ)^\circ \times G_1 \times \cdots \times G_d \to G^\circ \tag{6}$$

is a surjective group homomorphism with finite central kernel. The number d is not specified in advance, it indicates the number of independent variables in our upcoming Hecke algebras. Of course there are in general many ways to achieve (6). Two choices are always canonical:

(7)
- $G_1 = G^\circ_{\mathrm{der}}$, with $d = 1$;
- every G_j is of the form $N_1 N_2 \cdots N_k$, where $\{N_1, \ldots, N_k\}$ is a $N_G(q\mathcal{E})$-orbit of simple normal subgroups of G°.

In any case, (6) gives a decomposition

$$\mathfrak{g} = \mathfrak{g}_z \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_d \quad \text{where } \mathfrak{g}_z = \mathrm{Lie}(Z(G^\circ)), \mathfrak{g}_j = \mathrm{Lie}(G_j). \tag{8}$$

Each root system

$$R_j := R(G_j T, T) = R(G_j, G_j \cap T)$$

is a $W_{q\mathcal{E}}$-stable union of irreducible components of $R(G^\circ, T)$. Thus we obtain an orthogonal, $W_{q\mathcal{E}}$-stable decomposition

$$R(G^\circ, T) = R_1 \sqcup \cdots \sqcup R_d. \tag{9}$$

We let $\vec{\mathbf{r}} = (\mathbf{r}_1, \ldots, \mathbf{r}_d)$ be an array of variables, corresponding to (6) and (9) in the sense that $\mathbf{r}_j$ is relevant for G_j and R_j only. We abbreviate

$$\mathbb{C}[\vec{\mathbf{r}}] = \mathbb{C}[\mathbf{r}_1, \ldots, \mathbf{r}_d].$$

Let $\natural : (W_\mathcal{E}/W^\circ_\mathcal{E})^2 \to \mathbb{C}^\times$ be a 2-cocycle. Recall that the twisted group algebra $\mathbb{C}[W_\mathcal{E}, \natural]$ has a $\mathbb{C}$-basis $\{N_w : w \in W_\mathcal{E}\}$ and multiplication rules

$$N_w \cdot N_{w'} = \natural(w, w') N_{ww'}.$$

In particular it contains the group algebra of $W^\circ_\mathcal{E}$.

Let $c : R(G^\circ, T)_{\mathrm{red}} \to \mathbb{C}$ be a $W_{q\mathcal{E}}$-invariant function.

Proposition 1.1. *There exists a unique associative algebra structure on* $\mathbb{C}[W_{q\mathcal{E}}, \natural] \otimes S(\mathfrak{t}^*) \otimes \mathbb{C}[\vec{\mathbf{r}}]$ *such that:*

- *the twisted group algebra* $\mathbb{C}[W_{q\mathcal{E}}, \natural]$ *is embedded as subalgebra;*
- *the algebra* $S(\mathfrak{t}^*) \otimes \mathbb{C}[\vec{\mathbf{r}}]$ *of polynomial functions on* $\mathfrak{t} \oplus \mathbb{C}^d$ *is embedded as a subalgebra;*
- $\mathbb{C}[\vec{\mathbf{r}}]$ *is central;*
- *the braid relation* $N_{s_\alpha} \xi - {}^{s_\alpha}\xi N_{s_\alpha} = c(\alpha) \mathbf{r}_j (\xi - {}^{s_\alpha}\xi)/\alpha$ *holds for all* $\xi \in S(\mathfrak{t}^*)$ *and all simple roots* $\alpha \in R_j$
- $N_w \xi N_w^{-1} = {}^w\xi$ *for all* $\xi \in S(\mathfrak{t}^*)$ *and* $w \in \mathfrak{R}_{q\mathcal{E}}$.

Proof. For $d = 1, G_1 = G^\circ_{\mathrm{der}}$ this is [AMS2, Proposition 2.2]. The general case can be shown in the same way. □

We denote the algebra just constructed by $\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, c\vec{\mathbf{r}}, \natural)$. When $W^\circ_{q\mathcal{E}} = W_{q\mathcal{E}}$, there is no 2-cocycle, and write simply $\mathbb{H}(\mathfrak{t}, W^\circ_{q\mathcal{E}}, c\vec{\mathbf{r}})$. It is clear from the defining relations that

$$S(\mathfrak{t}^*)^{W_{q\mathcal{E}}} \otimes \mathbb{C}[\vec{\mathbf{r}}] = \mathcal{O}(\mathfrak{t} \times \mathbb{C}^d)^{W_{q\mathcal{E}}} \text{ is a central subalgebra of } \mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, c\vec{\mathbf{r}}, \natural). \tag{10}$$

To the cuspidal quasi-support $[M, \mathcal{C}^M_v, q\mathcal{E}]_G$ we associated a particular 2-cocycle

$$\natural_{q\mathcal{E}} : (W_{q\mathcal{E}}/W^\circ_{q\mathcal{E}})^2 \to \mathbb{C}^\times,$$

see [AMS1, Lemma 5.3]. The pair (M°, v) also gives rise to a $W_{q\mathcal{E}}$-invariant function $c : R(G^\circ, T)_{\mathrm{red}} \to \mathbb{Z}$, see [Lus2, Proposition 2.10] or [AMS2, (12)]. We denote the algebra $\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, c\vec{\mathbf{r}}, \natural_{q\mathcal{E}})$, with this particular c, by $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$.

In [AMS2] we only studied the case $d = 1, R_1 = R(G^\circ, T)$, and we denoted that algebra by $\mathbb{H}(G, M, q\mathcal{E})$. Fortunately the difference with $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$ is so small that almost all properties of $\mathbb{H}(G, M, q\mathcal{E})$ discussed in [AMS2] remain valid for $\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, c\vec{\mathbf{r}}, \natural_{q\mathcal{E}})$. We will proceed to make this precise.

Write $v = v_1 + \cdots + v_d$ with $v_j \in \mathfrak{g}_j = \mathrm{Lie}(G_j)$. Then

$$\mathcal{C}_v^{M^\circ} = \mathcal{C}_{v_1}^{M_1} + \cdots + \mathcal{C}_{v_d}^{M_d} \text{ , where } M_j = M^\circ \cap G_j .$$

The M°-action on $(\mathcal{C}_v^{M^\circ}, \mathcal{E})$ can be inflated to $Z(G^\circ)^\circ \times M_1 \times \cdots \times M_d$, and the pullback of $\mathcal{E}$ becomes trivial on $Z(G^\circ)^\circ$ and decomposes uniquely as

$$m_{G^\circ}^* \mathcal{E} = \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_d \tag{11}$$

with $\mathcal{E}_j$ a M_j-equivariant cuspidal local system on $\mathcal{C}_{v_j}^{M_j}$. From Proposition 1.1 and [AMS2, Proposition 2.2] we see that

$$\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}}) = \mathbb{H}(G_1, M_1, \mathcal{E}_1) \otimes \cdots \otimes \mathbb{H}(G_d, M_d, \mathcal{E}_d). \tag{12}$$

Furthermore the proof of [AMS2, Proposition 2.2] shows that

$$\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}) = \mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}}) \rtimes \mathbb{C}[\mathfrak{R}_{q\mathcal{E}}, \natural_{q\mathcal{E}}]. \tag{13}$$

To parametrize the irreducible representations of these algebras we use some elements of the Lie algebras of the involved algebraic groups. Let $\sigma_0 \in \mathfrak{g}$ be semisimple and $y \in Z_{\mathfrak{g}}(\sigma_0)$ be nilpotent. We decompose them along (8):

$$\begin{aligned} \sigma_0 &= \sigma_z + \sigma_{0,1} + \cdots + \sigma_{0,d} \quad \text{with } \sigma_{0,j} \in \mathfrak{g}_j, \sigma_z \in \mathfrak{g}_z, \\ y &= y_1 + \cdots + y_d \quad \text{with } y_j \in \mathfrak{g}_j. \end{aligned}$$

Choose algebraic homomorphisms $\gamma_j : \mathrm{SL}_2(\mathbb{C}) \to G_j$ with $\mathrm{d}\gamma_j \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = y_j$. Given $\vec{r} \in \mathbb{C}^d$, we write $\sigma_j = \sigma_{0,j} + \mathrm{d}\gamma_j \left(\begin{smallmatrix} r_j & 0 \\ 0 & -r_j \end{smallmatrix}\right)$ and

$$\begin{aligned} \mathrm{d}\vec{\gamma} \left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) &= \mathrm{d}\gamma_1 \left(\begin{smallmatrix} r_1 & 0 \\ 0 & -r_1 \end{smallmatrix}\right) + \cdots + \mathrm{d}\gamma_d \left(\begin{smallmatrix} r_d & 0 \\ 0 & -r_d \end{smallmatrix}\right), \\ \sigma &= \sigma_0 + \mathrm{d}\vec{\gamma} \left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right). \end{aligned} \tag{14}$$

Notice that $[\sigma, y_j] = [\sigma_j, y_j] = 2 r_j y_j$. Let us recall the construction of the standard modules from [Lus2, AMS2]. We need the groups

$$\begin{aligned} M_j(y_j) &= \big\{ (g_j, \lambda_j) \in G_j \times \mathbb{C}^\times : \mathrm{Ad}(g_j) y_j = \lambda_j^2 y_j \big\}, \\ \vec{M}^\circ(y) &= \big\{ (g, \vec{\lambda}) \in G^\circ \times (\mathbb{C}^\times)^d : \mathrm{Ad}(g) y_j = \lambda_j^2 y_j \; \forall j = 1, \ldots, d \big\}, \\ \vec{M}(y) &= \big\{ (g, \vec{\lambda}) \in G^\circ N_G(q\mathcal{E}) \times (\mathbb{C}^\times)^d : \mathrm{Ad}(g) y_j = \lambda_j^2 y_j \; \forall j = 1, \ldots, d \big\}, \end{aligned}$$

and the varieties

$$\begin{aligned} \mathcal{P}_{y_j} &= \big\{ g(P^\circ \cap G_j) \in G_j / (P^\circ \cap G_j) : \mathrm{Ad}(g^{-1}) y_j \in \mathcal{C}_{v_j}^{M_j} + \mathrm{Lie}(U \cap G_j) \big\}, \\ \mathcal{P}_y^\circ &= \big\{ g P^\circ \in G^\circ / P^\circ : \mathrm{Ad}(g^{-1}) y \in \mathcal{C}_v^{M^\circ} + \mathrm{Lie}(U) \big\}, \\ \mathcal{P}_y &= \big\{ g P \in G^\circ N_G(q\mathcal{E}) / P : \mathrm{Ad}(g^{-1}) y \in \mathcal{C}_v^M + \mathrm{Lie}(U) \big\}. \end{aligned}$$

The local systems $\mathcal{E}_j, \mathcal{E}$ and $q\mathcal{E}$ give rise to local systems $\dot{\mathcal{E}}_j, \dot{\mathcal{E}}$ and $\dot{q\mathcal{E}}$ on $\mathcal{P}_{y_j}, \mathcal{P}_y^\circ$ and $\mathcal{P}_y$, respectively. The groups $M_j(y_j), \vec{M}^\circ(y)$ and $\vec{M}(y)$ act naturally on, respectively, $(\mathcal{P}_{y_j}, \dot{\mathcal{E}}_j), (\mathcal{P}_y^\circ, \dot{\mathcal{E}})$ and $(\mathcal{P}_y, \dot{q\mathcal{E}})$. With the method from [Lus2] and [AMS2,

§3.1] we can define an action of $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}) \times \vec{M}(y)$ on the equivariant homology $H_*^{\vec{M}(y)^\circ}(\mathcal{P}_y, \dot{q\mathcal{E}})$, and similarly for $H_*^{\vec{M}^\circ(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{E}})$ and $H_*^{M_j(y)^\circ}(\mathcal{P}_{y_j}, \dot{\mathcal{E}}_j)$. As in [Lus2] we build

$$E^\circ_{y_j,\sigma_j,r_j} = \mathbb{C}_{\sigma_j,r_j} \underset{H^*_{M_j(y_j)^\circ}(\{y_j\})}{\otimes} H_*^{M_j(y)^\circ}(\mathcal{P}_{y_j}, \dot{\mathcal{E}}_j).$$

Similarly we introduce

$$E^\circ_{y,\sigma,\vec{r}} = \mathbb{C}_{\sigma,\vec{r}} \underset{H^*_{\vec{M}^\circ(y)^\circ}(\{y\})}{\otimes} H_*^{\vec{M}^\circ(y)^\circ}(\mathcal{P}^\circ_y, \dot{\mathcal{E}}),$$

$$E_{y,\sigma,\vec{r}} = \mathbb{C}_{\sigma,\vec{r}} \underset{H^*_{\vec{M}(y)^\circ}(\{y\})}{\otimes} H_*^{\vec{M}(y)^\circ}(\mathcal{P}_y, \dot{q\mathcal{E}}).$$

By [AMS2, Theorem 3.2 and Lemma 3.6] these are modules over, respectively, $\mathbb{H}(G_j, M_j, \mathcal{E}_j) \times \pi_0(Z_{G_j}(\sigma_{0,j}, y_j))$, $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}}) \times \pi_0(Z_{G^\circ}(\sigma_0, y))$ and $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}) \times \pi_0(Z_{G^\circ N_G(q\mathcal{E})}(\sigma_0, y))$. This last action is the reason to use $G^\circ N_G(q\mathcal{E})$ instead of G in the definition of $\mathcal{P}_y$.

In terms of (13), there is a natural module isomorphism

$$E_{y,\sigma,\vec{r}} \cong \mathrm{ind}_{\mathbb{H}(G^\circ,M^\circ,\mathcal{E},\vec{\mathbf{r}})}^{\mathbb{H}(G,M,q\mathcal{E},\vec{\mathbf{r}})} E^\circ_{y,\sigma,\vec{r}}. \tag{15}$$

It can be proven in the same way as the analogous statement with only one variable $\mathbf{r}$, which is [AMS2, Lemma 3.3].

Lemma 1.2. *With the identifications* (12) *there is a natural isomorphism of* $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$*-modules*

$$E^\circ_{y,\sigma,\vec{r}} \cong \mathbb{C}_{\sigma_z} \otimes E^\circ_{y_1,\sigma_1,r_1} \otimes \cdots \otimes E^\circ_{y_d,\sigma_d,r_d},$$

which is equivariant for the actions of the appropriate subquotients of $\vec{M}^\circ(y)$.

Proof. From (6) and $Z(G^\circ)Z(G_j) \subset P^\circ$ we get natural isomorphisms

$$\mathcal{P}_{y_1} \times \cdots \times \mathcal{P}_{y_d} \to \mathcal{P}^\circ_y. \tag{16}$$

Looking at (11) and the construction of $\dot{\mathcal{E}}$ in [Lus2, §3.4], we deduce that

$$\dot{\mathcal{E}} \cong \dot{\mathcal{E}}_1 \otimes \cdots \otimes \dot{\mathcal{E}}_d \text{ as sheaves on } \mathcal{P}^\circ_y. \tag{17}$$

From (6) we also get a central extension

$$1 \to \ker m_{G^\circ} \to Z(G^\circ)^\circ \times M_1(y_1) \times \cdots \times M_d(y_d) \to \vec{M}^\circ(y) \to 1. \tag{18}$$

Here $\ker m_{G^\circ}$ refers to the kernel of (6), a finite central subgroup which acts trivially on the sheaf $\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_d \cong m^*_{G^\circ}\mathcal{E}$. Restricting to connected components, we obtain a central extension of $\vec{M}^\circ(y)^\circ$ by

$$\tilde{M} := Z(G^\circ)^\circ \times M_1(y_1)^\circ \times \cdots \times M_d(y_d)^\circ$$

In fact, equivariant (co)homology is inert under finite central extensions, for all groups and all varieties. We sketch how this can be deduced from [Lus2, §1]. By definition

$$H^*_{\vec{M}^\circ(y)^\circ}(\mathcal{P}^\circ_y, \dot{\mathcal{E}}) = H^*\big(\vec{M}^\circ(y)^\circ \backslash (\Gamma \times \mathcal{P}^\circ_y), {}_\Gamma\dot{\mathcal{E}}\big)$$

for a suitable (in particular free) $\vec{M}^\circ(y)^\circ$-variety Γ and a local system derived from $\dot{\mathcal{E}}$. On the right hand side we can replace $\vec{M}^\circ(y)^\circ$ by $\tilde{M}$ without changing anything. If $\tilde{\Gamma}$ is a suitable variety for $\tilde{M}$, then $\tilde{\Gamma} \times \Gamma$ is also one. (The freeness is preserved because (18) is an extension of finite index.) The argument in [Lus2, p. 149] shows that

$$H^*\big(\tilde M \backslash (\Gamma \times \mathcal{P}_y^\circ), {}_\Gamma \dot{\mathcal{E}}\big) \cong H^*\big(\tilde M \backslash (\Gamma' \times \Gamma \times \mathcal{P}_y^\circ), {}_{\Gamma' \times \Gamma} \dot{\mathcal{E}}\big) = H^*_{\tilde M}(\mathcal{P}_y^\circ, \dot{\mathcal{E}}).$$

In a similar way, using [Lus2, Lemma 1.2], one can prove that

$$H_*^{\vec{M}^\circ(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{E}}) \cong H_*^{\tilde M}(\mathcal{P}_y^\circ, \dot{\mathcal{E}}). \tag{19}$$

The upshot of (16), (17) and (19) is that we can factorize the entire setting along (12), which gives

$$H_*^{M_1(y)^\circ}(\mathcal{P}_{y_1}, \dot{\mathcal{E}}_1) \otimes \cdots \otimes H_*^{M_d(y)^\circ}(\mathcal{P}_{y_d}, \dot{\mathcal{E}}_d) \cong H_*^{\vec{M}^\circ(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{E}}). \tag{20}$$

The equivariant cohomology of a point with respect to a connected group depends only on the Lie algebra [Lus2, §1.11], so (18) implies a natural isomorphism

$$H^*_{Z(G^\circ)^\circ}(\{1\}) \times H^*_{M_1(y_1)^\circ}(\{y_1\}) \times \cdots \times H^*_{M_d(y_d)^\circ}(\{y_d\}) \cong H^*_{\vec{M}^\circ(y)^\circ}(\{y\}).$$

Thus we can tensor both sides of (20) with $\mathbb{C}_{\sigma,\vec r}$ and preserve the isomorphism. □

Given $\rho_j \in \mathrm{Irr}\big(\pi_0(Z_{G_j}(\sigma_{0,j}, y_j))\big)$, we can form the standard $\mathbb{H}(G_j, M_j, \mathcal{E}_j)$-module

$$E^\circ_{y_j,\sigma_j,r_j,\rho_j} := \mathrm{Hom}_{\pi_0(Z_{G_j}(\sigma_{0,j},y_j))}(\rho_j, E^\circ_{y_j,\sigma_j,r_j}).$$

Similarly $\rho^\circ \in \mathrm{Irr}\big(\pi_0(Z_{G^\circ}(\sigma_0, y))\big)$ and $\rho \in \mathrm{Irr}\big(\pi_0(Z_{G^\circ N_G(q\mathcal{E})}(\sigma_0, y))\big)$ give rise to

$$\begin{aligned} E^\circ_{y,\sigma,\vec r,\rho^\circ} &:= \mathrm{Hom}_{\pi_0(Z_{G^\circ}(\sigma_0,y))}(\rho^\circ, E^\circ_{y,\sigma,\vec r}), \\ E_{y,\sigma,\vec r,\rho} &:= \mathrm{Hom}_{\pi_0(Z_{G^\circ N_G(q\mathcal{E})}(\sigma_0,y))}(\rho, E_{y,\sigma,\vec r}). \end{aligned} \tag{21}$$

We call these standard modules for respectively $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$ and $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$.

The canonical map (6) induces a surjection

$$\pi_0(Z_{G_1}(\sigma_{0,1}, y_1)) \times \cdots \times \pi_0(Z_{G_d}(\sigma_{0,d}, y_d)) \to \pi_0(Z_{G^\circ}(\sigma_0, y)). \tag{22}$$

Lemma 1.3. *Let $\rho^\circ \in \mathrm{Irr}\big(\pi_0(Z_{G^\circ}(\sigma_0, y))\big)$ and let $\bigotimes_{j=1}^d \rho_j$ be its inflation to $\prod_{j=1}^d \pi_0(Z_{G_j}(\sigma_{0,j}, y_j))$ via (22). There is a natural isomorphism of $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$-modules*

$$E^\circ_{y,\sigma,\vec r,\rho^\circ} \cong \mathbb{C}_{\sigma_z} \otimes E^\circ_{y_1,\sigma_1,r_1,\rho_1} \otimes \cdots \otimes E^\circ_{y_d,\sigma_d,r_d,\rho_d}.$$

Every $\bigotimes_{j=1}^d \rho_j \in \mathrm{Irr}\big(\prod_{j=1}^d \pi_0(Z_{G_j}(\sigma_{0,j}, y_j))\big)$ for which $\bigotimes_{j=1}^d E^\circ_{y_j,\sigma_j,r_j,\rho_j}$ is nonzero comes from $\pi_0(Z_{G^\circ}(\sigma_0, y))$ via (22).

Proof. The module isomorphism follows from the naturality and the equivariance in Lemma 1.2.

Suppose that $\bigotimes_{j=1}^d \rho_j \in \mathrm{Irr}\big(\prod_{j=1}^d \pi_0(Z_{G_j}(\sigma_{0,j}, y_j))\big)$ appears in $\bigotimes_{j=1}^d E^\circ_{y_j,\sigma_j,r_j}$. By [AMS2, Proposition 3.7] the cuspidal support $\Psi_{Z_G(\sigma_{0,j})}(y_j, \rho_j)$ is G_j-conjugate to $(M_j, \mathcal{C}^{M_j}_{y_j}, \mathcal{E}_j)$. In particular ρ_j has the same $Z(G_j)$-character as $\mathcal{E}_j$, see [Lus1, Theorem 6.5.a]. Hence $\otimes_j \rho_j$ has the same central character as $m^*_{G_0}\mathcal{E}$. That central character factors through the multiplication map (6) whose kernel is central, so $\otimes_j \rho_j$ also factors through (6). That is, the map (22) induces a bijection between the relevant irreducible representations on both sides. □

For some choices of ρ the standard module $E_{y,\sigma,\vec r,\rho}$ is zero. To avoid that, we consider triples (σ_0, y, ρ) with

- $\sigma_0 \in \mathfrak{g}$ is semisimple,

- $y \in Z_{\mathfrak g}(\sigma_0)$ is nilpotent,
- $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma_0, y))\big)$ is such that the cuspidal quasi-support $q\Psi_{Z_G(\sigma_0)}(y,\rho)$ from [AMS1, §5] is G-conjugate to $(M, \mathcal C_v^M, q\mathcal E)$.

Given in addition $\vec r \in \mathbb C^d$, we construct $\sigma = \sigma_0 + \mathrm d \vec\gamma \left(\begin{smallmatrix} \vec r & 0 \\ 0 & -\vec r \end{smallmatrix}\right) \in \mathfrak g$ as in (14). Although this depends on the choice of $\vec\gamma$, the conjugacy class of σ does not.

By definition

$$\mathbb H(G^\circ N_G(q\mathcal E), M, q\mathcal E, \vec{\mathbf r}) = \mathbb H(G, M, q\mathcal E, \vec{\mathbf r}),$$

but of course $\pi_0(Z_{G^\circ N_G(q\mathcal E)}(\sigma_0, y))$ can be a proper subgroup of $\pi_0(Z_G(\sigma_0,y))$. As shown in the proof of [AMS2, Lemma 3.21], the functor $\mathrm{ind}_{\pi_0(Z_{G^\circ N_G(q\mathcal E)}(\sigma_0,y))}^{\pi_0(Z_G(\sigma_0,y))}$ provides a bijection between the $\tilde\rho$ in the triples for $G^\circ N_G(q\mathcal E)$ and the ρ in the triples for G. For $\rho = \mathrm{ind}_{\pi_0(Z_{G^\circ N_G(q\mathcal E)}(\sigma_0,y))}^{\pi_0(Z_G(\sigma_0,y))} \tilde\rho$ we define, in terms of (21),

$$E_{y,\sigma,\vec r,\rho} = E_{y,\sigma,\vec r,\tilde\rho}. \tag{23}$$

The next result generalizes [AMS2, Theorem 3.20] to several variables r_j. We define $\mathrm{Irr}_{\vec r}(\mathbb H(G, M, q\mathcal E, \vec{\mathbf r}))$ to be the set of equivalence classes of those irreducible representations of $\mathbb H(G, M, q\mathcal E, \vec{\mathbf r}$ on which $\mathbf r_j$ acts as r_j.

Theorem 1.4. *Fix $\vec r \in \mathbb C^d$. The standard $\mathbb H(G, M, q\mathcal E, \vec{\mathbf r})$-module $E_{y,\sigma,\vec r,\rho}$ is nonzero if and only if $q\Psi_{Z_G(\sigma_0)}(y,\rho) = (M, \mathcal C_v^M, q\mathcal E)$ up to G-conjugacy. In that case it has a distinguished irreducible quotient $M_{y,\sigma,\vec r,\rho}$, which appears with multiplicity one in $E_{y,\sigma,\vec r,\rho}$.*

The map $M_{y,\sigma,\vec r,\rho} \longleftrightarrow (\sigma_0, y, \rho)$ sets up a canonical bijection between $\mathrm{Irr}_{\vec r}(\mathbb H(G, M, q\mathcal E, \vec{\mathbf r}))$ and G-conjugacy classes of triples as above.

Proof. For $\mathbb H(G_j, M_j, \mathcal E_j)$ this is [AMS2, Proposition 3.7 and Theorem 3.11]. With (12) and Lemma 1.3 we can generalize that to $\mathbb H(G^\circ, M^\circ, q\mathcal E, \vec{\mathbf r})$. The method to go from there to $\mathbb H(G^\circ N_G(q\mathcal E), M, q\mathcal E, \vec{\mathbf r})$ is exactly the same as in [AMS2, §3.3–3.4] (for $\mathbb H(G^\circ, M^\circ, \mathcal E)$ and $\mathbb H(G^\circ N_G(q\mathcal E), M, q\mathcal E)$). That is, the proof of [AMS2, Theorem 3.20] applies and establishes the theorem for $\mathbb H(G^\circ N_G(q\mathcal E), M, q\mathcal E, \vec{\mathbf r})$. In view of (23) we can replace $G^\circ N_G(q\mathcal E)$ by G. □

The above modules are compatible with parabolic induction, in a suitable sense. Let $Q \subset G$ be an algebraic subgroup such that $Q \cap G^\circ$ is a Levi subgroup of G° and $L \subset Q^\circ$. Let y, σ, r, ρ be as in Theorem 1.4, with $\sigma, y \in \mathfrak q = \mathrm{Lie}(Q)$. By [Ree, §3.2] the natural map

$$\pi_0(Z_Q(\sigma,y)) = \pi_0(Z_{Q \cap Z_G(\sigma_0)}(y)) \to \pi_0(Z_{Z_G(\sigma_0)}(y)) = \pi_0(Z_G(\sigma,y)) \tag{24}$$

is injective, so we can consider the left hand side as a subgroup of the right hand side. Let $\rho^Q \in \mathrm{Irr}\big(\pi_0(Z_Q(\sigma,y))\big)$ be such that $q\Psi_{Z_Q(\sigma_0)}(y,\rho^Q) = (M, \mathcal C_v^M, q\mathcal E)$. Then $E_{y,\sigma,r,\rho}, M_{y,\sigma,r,\rho}, E^Q_{y,\sigma,r,\rho^Q}$ and M^Q_{y,σ,r,ρ^Q} are defined.

Proposition 1.5. (a) *There is a natural isomorphism of $\mathbb H(G, M, q\mathcal E, \vec{\mathbf r})$-modules*

$$\mathbb H(G, M, q\mathcal E, \vec{\mathbf r}) \underset{\mathbb H(Q,M,q\mathcal E,\vec{\mathbf r})}{\otimes} E^Q_{y,\sigma,\vec r,\rho^Q} \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho) \otimes E_{y,\sigma,r,\rho},$$

where the sum runs over all $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma,y))\big)$ with $q\Psi_{Z_G(\sigma_0)}(y,\rho) = (M, \mathcal C_v^M, q\mathcal E)$.

(*b*) *For* $\vec{r} = \vec{0}$ *part (a) contains an isomorphism of* $S(\mathfrak{t}^*) \rtimes \mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$*-modules*

$$\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}) \underset{\mathbb{H}(Q,M,q\mathcal{E},\vec{\mathbf{r}})}{\otimes} M^Q_{y,\sigma,\vec{0},\rho^Q} \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho) \otimes M_{y,\sigma,\vec{0},\rho}.$$

(*c*) *The multiplicity of* $M_{y,\sigma,\vec{r},\rho}$ *in* $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}) \underset{\mathbb{H}(Q,M,q\mathcal{E},\vec{\mathbf{r}})}{\otimes} E^Q_{y,\sigma,\vec{r},\rho^Q}$ *is* $[\rho^Q : \rho]_{\pi_0(Z_Q(\sigma,y))}$. *It already appears that many times as a quotient, via* $E^Q_{y,\sigma,\vec{r},\rho^Q} \to M^Q_{y,\sigma,\vec{r},\rho^Q}$. *More precisely, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbb{H}(Q,M,q\mathcal{E},\vec{\mathbf{r}})}(M^Q_{y,\sigma,\vec{r},\rho^Q}, M_{y,\sigma,\vec{r},\rho}) \cong \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho)^*.$$

Proof. For twisted graded Hecke algebras with only one parameter $\mathbf{r}$ this is [AMS2, Proposition 3.22]. Using Theorem 1.4, the proof of that result also works in the present setting. □

For an improved parametrization we use the Iwahori–Matsumoto involution, whose definition we will now generalize to $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$. Extend the sign representation of the Weyl group $W^\circ_{q\mathcal{E}}$ to a character of $W_{q\mathcal{E}}$ which is trivial on $\mathfrak{R}_{q\mathcal{E}}$. Then we define

$$\mathrm{IM}(N_w) = \mathrm{sign}(w) N_w,\ \mathrm{IM}(\mathbf{r}_j) = \mathbf{r}_j,\ \mathrm{IM}(\xi) = -\xi\ (\xi \in \mathfrak{t}^*). \tag{25}$$

Twisting representations with this involution is useful in relation with the properties temperedness and (essentially) discrete series, see [AMS2, §3.5].

Proposition 1.6. (*a*) *Fix* $\vec{r} \in \mathbb{C}^d$. *There exists a canonical bijection*

$$(\sigma_0, y, \rho) \longleftrightarrow \mathrm{IM}^* M_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho}$$

between conjugacy classes triples as in Theorem 1.4 and $\mathrm{Irr}_{\vec{r}}(\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}))$.

(*b*) *Suppose that* $\Re(\vec{r}) \in \mathbb{R}^d_{\geq 0}$. *Then* $\mathrm{IM}^* M_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho}$ *is tempered if and only if* $\sigma_0 \in i\mathfrak{t}_\mathbb{R} = i\mathbb{R} \otimes_\mathbb{Z} X_*(T)$.

(*c*) *Suppose that* $\Re(\vec{r}) \in \mathbb{R}^d_{>0}$. *Then* $\mathrm{IM}^* M_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho}$ *is essentially discrete series if and only if* y *is distinguished in* $\mathfrak{g}$.
Moreover, in this case $\sigma_0 = 0$ *and* $E_{y,-\sigma,\vec{r},\rho} = M_{y,-\sigma,\vec{r},\rho}$.

Proof. Part (a) follows immediately from Theorem 1.4. Parts (b) and (c) are consequences of [AMS2, §3.5], see in particular (84) and (85) therein. □

2. Twisted affine Hecke algebras

We would like to push the results of [AMS2] and the previous section to affine Hecke algebras, because these appear more directly in the representation theory of reductive p-adic groups. This can be achieved with Lusztig's reduction theorems [Lus3]. The first reduces to representations with a "real" central character (to be made precise later), and the second reduction theorem relates representations of affine Hecke algebras with representations of graded Hecke algebras.

Our goal is a little more specific though, we want to consider not just one (twisted) graded Hecke algebra, but a family of those, parametrized by a torus. We want to find a (twisted) affine Hecke algebra which contains all members of this family as some kind of specialization. Let us mention here that, although we phrase this section with quasi-Levi subgroups and cuspidal quasi-supports, all the results are equally valid for Levi subgroups and cuspidal supports.

Let G be a possibly disconnected complex reductive group and let $(M, \mathcal{C}_v^M, q\mathcal{E})$ be a cuspidal quasi-support for G. For any $t \in T = Z(M)^\circ$ the reductive group $G_t = Z_G(t)$ contains M, and we can consider the twisted graded Hecke algebra

$$\mathbb{H}(G_t, M, q\mathcal{E}, \vec{\mathbf{r}}) = \mathbb{H}(\mathfrak{t}, N_{G_t}(q\mathcal{E})/M, c_t\vec{\mathbf{r}}, \natural_{q\mathcal{E},t}).$$

Here $\vec{\mathbf{r}} = (\mathbf{r}_1, \ldots, \mathbf{r}_d)$ refers to the almost direct factorization of G_t° induced by (8). Let us investigate how these algebras depend on t. For any $t \in T$, the 2-cocycle $\natural_{q\mathcal{E},t}$ of $N_{G_t}(q\mathcal{E})/M$ is just the restriction of $\natural_{q\mathcal{E}} : W_{q\mathcal{E}}^2 \to \mathbb{C}^\times$. This can be seen from [Lus1, §3] and the proofs of [AMS1, Proposition 4.5 and Lemma 5.4]. More concretely, the perverse sheaves $(\mathrm{pr}_1)_! q\dot{\mathcal{E}}$ and $(\mathrm{pr}_1)_! q\dot{\mathcal{E}}^*$ on $\mathrm{Lie}(G)$ from [AMS2, (90)] and [Lus2, §3.4] naturally contain the corresponding objects $(\mathrm{pr}_{1,t})_! q\dot{\mathcal{E}}, (\mathrm{pr}_{1,t})_! q\dot{\mathcal{E}}^*$ for G_t. The algebra

$$\mathbb{C}[N_{G_t}(q\mathcal{E})/M, \natural_{q\mathcal{E},t}] \cong \mathrm{End}_{\mathcal{D}\mathrm{Lie}(G_t)_{\mathrm{RS}}}\big((\mathrm{pr}_{1,t})_! q\dot{\mathcal{E}}\big)$$

from [AMS1, Proposition 4.5] is canonically embedded in

$$\mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}] \cong \mathrm{End}_{\mathcal{D}\mathrm{Lie}(G)_{\mathrm{RS}}}\big((\mathrm{pr}_1)_! q\dot{\mathcal{E}}\big).$$

We will simply write $W_{q\mathcal{E},t}$ for $N_{G_t}(q\mathcal{E})/M$, and $\natural_{q\mathcal{E}}$ for $\natural_{q\mathcal{E},t}$.

On the other hand, the parameter function $c_t : R(Z_G(t)^\circ, T)_{\mathrm{red}} \to \mathbb{C}$ can depend on t. Recall that $c_t(\alpha)$ was defined in [Lus2, §2]. For any root $\alpha \in R(G^\circ, T)$:

$$\mathfrak{g}_\alpha \subset \mathrm{Lie}(G_t) \iff \alpha(t) = 1.$$

From [Lus2, Proposition 2.2] we know that $R(G^\circ, T)$ is a root system, so $R(G^\circ, T) \cap \mathbb{R}\alpha \subset \{a, 2\alpha, -\alpha, -2\alpha\}$ for every nondivisible root α.

Proposition 2.1. [Lus2, Propositions 2.8 and 2.10]

(*a*) *Suppose that $R(G^\circ, T) \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$. Then $c_t(\alpha)$ satisfies*

$$0 = \mathrm{ad}(v)^{c_t(\alpha)-1} : \mathfrak{g}_\alpha \to \mathfrak{g}_\alpha \quad \textit{and} \quad 0 \neq \mathrm{ad}(v)^{c_t(\alpha)-2} : \mathfrak{g}_\alpha \to \mathfrak{g}_\alpha. \tag{26}$$

This condition is independent of t, as long as $\mathfrak{g}_\alpha \subset \mathrm{Lie}(G_t)$. So we can unambiguously write $c(\alpha)$ for $c_t(\alpha)$ in this case. Moreover $c(\alpha) \in \mathbb{N}$ is even.

(*b*) *Suppose that $R(G^\circ, T) \cap \mathbb{R}\alpha = \{a, 2\alpha, -\alpha, -2\alpha\}$.*

When $\alpha(t) = 1$, $\{\alpha, 2\alpha\} \subset R(Z_G(t)^\circ, T)$. Then $c_t(\alpha)$ is again given by (26)*, and it is odd. We write $c(\alpha) = c_t(\alpha)$ for such a $t \in T$. Furthermore $c_t(2\alpha)$ is given by* (26) *with 2α instead of α, and it equals 2.*

When $\alpha(t) = -1$, still $2\alpha \in R(Z_G(t)^\circ, T)$, and $c_t(2\alpha)$ is given by (26) *with 2α instead of α. It equals 2, and we write $c(2\alpha) = 2$.*

With the conventions from Proposition 2.1, c_t is always the restriction of $c : R(G^\circ, T) \to \mathbb{C}$ to $R(Z_G(t)^\circ, T)_{\mathrm{red}}$.

Now we construct the algebras that we need.

Proposition 2.2. *Consider the following data:*

- *the root datum $\mathcal{R} = (R(G^\circ, T), X^*(T), R(G^\circ, T)^\vee, X_*(T))$, with simple roots determined by P;*
- *the group $W_{q\mathcal{E}} = W_{\mathcal{E}}^\circ \rtimes \mathfrak{R}_{q\mathcal{E}}$;*
- *a 2-cocycle $\natural : (W_{q\mathcal{E}}/W_{\mathcal{E}}^\circ)^2 \to \mathbb{C}^\times$;*
- *$W_{q\mathcal{E}}$-invariant functions $\lambda : R(G^\circ, T)_{\mathrm{red}} \to \mathbb{Z}_{\geq 0}$ and $\lambda^* : \{\alpha \in R(G^\circ, T)_{\mathrm{red}} : \alpha^\vee \in 2X_*(T)\} \to \mathbb{Z}_{\geq 0}$;*

- *an array of invertible variables* $\vec{\mathbf{z}} = (\mathbf{z}_1, \ldots, \mathbf{z}_d)$*, corresponding to the decomposition* (8) *of* $\mathfrak{g}$.

The vector space

$$\mathcal{O}(T \times (\mathbb{C}^\times)^d) \otimes \mathbb{C}[W_{q\mathcal{E}}] = \mathbb{C}[X^*(T)] \otimes \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}] \otimes \mathbb{C}[W_\mathcal{E}^\circ] \otimes \mathbb{C}[\mathfrak{R}_{q\mathcal{E}}, \natural]$$

admits a unique algebra structure such that:

- $\mathbb{C}[X^*(T)], \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}]$ *and* $\mathbb{C}[\mathfrak{R}_{q\mathcal{E}}, \natural]$ *are embedded as subalgebras;*
- $\mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}] = \mathbb{C}[\mathbf{z}_1, \mathbf{z}_1^{-1}, \ldots, \mathbf{z}_d, \mathbf{z}_d^{-1}]$ *is central;*
- *the span of* $W_\mathcal{E}^\circ$ *is the Iwahori–Hecke algebra* $\mathcal{H}(W_\mathcal{E}^\circ, \vec{\mathbf{z}}^{2\lambda})$ *of* $W_\mathcal{E}^\circ$ *with parameters* $\vec{\mathbf{z}}^{2\lambda(\alpha)}$. *That is, it has a basis* $\{N_w : w \in W_\mathcal{E}^\circ\}$ *such that*

$$\begin{array}{rcll} N_w N_v & = & N_{wv} & \text{if } \ell(w) + \ell(v) = \ell(wv), \\ (N_{s_\alpha} + \mathbf{z}_j^{-\lambda(\alpha)})(N_{s_\alpha} - \mathbf{z}_j^{\lambda(\alpha)}) & = & 0 & \text{if } \alpha \in R(G_j T, T)_{\mathrm{red}} \text{ is a simple root.} \end{array}$$

- *for* $\gamma \in \mathfrak{R}_{q\mathcal{E}}, w \in W_\mathcal{E}^\circ$ *and* $x \in X^*(T)$*:*

$$N_\gamma N_w \theta_x N_\gamma^{-1} = N_{\gamma w \gamma^{-1}} \theta_{\gamma(x)}.$$

- *for simple root* $\alpha \in R(G_j T, T)_{\mathrm{red}}$ *and* $x \in X^*(T)$*, corresponding to* $\theta_x \in \mathcal{O}(T)$*:*

$$\theta_x N_{s_\alpha} - N_{s_\alpha} \theta_{s_\alpha(x)} =$$
$$\begin{cases} \big(\mathbf{z}_j^{\lambda(\alpha)} - \mathbf{z}_j^{-\lambda(\alpha)}\big)(\theta_x - \theta_{s_\alpha(x)})/(\theta_0 - \theta_{-\alpha}) & \alpha^\vee \notin 2X_*(T) \\ \big(\mathbf{z}_j^{\lambda(\alpha)} - \mathbf{z}_j^{-\lambda(\alpha)} + \theta_{-\alpha}(\mathbf{z}_j^{\lambda^*(\alpha)} - \mathbf{z}_j^{-\lambda^*(\alpha)})\big)(\theta_x - \theta_{s_\alpha(x)})/(\theta_0 - \theta_{-2\alpha}) & \alpha^\vee \in 2X_*(T) \end{cases}$$

Proof. In the case $\mathfrak{R}_{q\mathcal{E}} = 1$, the existence and uniqueness of such an algebra is well-known. It follows for instance from [Lus3, §3], once we identify T_{s_α} from [Lus3] with $\mathbf{z}_j^{\lambda(\alpha)} N_{s_\alpha}$. It is called an affine Hecke algebra and denoted by $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{\mathbf{z}})$.

Since λ and λ^* are $W_{q\mathcal{E}}$-invariant,

$$A_\gamma : N_w \theta_x \mapsto N_{\gamma w \gamma^{-1}} \theta_{\gamma(x)}$$

defines an automorphism of $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{\mathbf{z}})$. Clearly

$$\mathfrak{R}_{q\mathcal{E}} \to \mathrm{Aut}(\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{\mathbf{z}})) : \gamma \mapsto A_\gamma$$

is a group homomorphism. Pick a central extension $\mathfrak{R}_{q\mathcal{E}}^+$ of $\mathfrak{R}_{q\mathcal{E}}$ and a central idempotent $p_\natural$ such that $\mathbb{C}[\mathfrak{R}_{q\mathcal{E}}, \natural] \cong p_\natural \mathbb{C}[\mathfrak{R}_{q\mathcal{E}}^+]$. Now the same argument as in the proof of [AMS2, Proposition 2.2] shows that the algebra

$$\mathbb{C}[\mathfrak{R}_{q\mathcal{E}}, \natural] \ltimes \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{\mathbf{z}}) \cong p_\natural \mathbb{C}[\mathfrak{R}_{q\mathcal{E}}^+] \ltimes \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{\mathbf{z}}) \subset \mathfrak{R}_{q\mathcal{E}}^+ \ltimes \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{\mathbf{z}}) \tag{27}$$

has the required properties. □

When $\mathfrak{R}_{q\mathcal{E}} = 1$, specializations of $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{\mathbf{z}})$ at $\vec{\mathbf{r}} = \vec{r} \in \mathbb{R}_{>0}^d$ figure for example in [Opd1]. In relation with p-adic groups one should think of the variables $\vec{\mathbf{z}}$ as as $(q_j^{1/2})_{j=1}^d$, where q_j is the cardinality of some finite field.

We define, for $\alpha \in R(G^\circ, T)_{\mathrm{red}}$:

$$\begin{array}{rcll} \lambda(\alpha) & = & c(\alpha)/2 & 2\alpha \notin R(G^\circ, T) \\ \lambda^*(\alpha) & = & c(\alpha)/2 & 2\alpha \notin R(G^\circ, T), \alpha^\vee \in 2X_*(T) \\ \lambda(\alpha) & = & c(\alpha)/2 + c(2\alpha)/4 & 2\alpha \in R(G^\circ, T) \\ \lambda^*(\alpha) & = & c(\alpha)/2 - c(2\alpha)/4 & 2\alpha \in R(G^\circ, T). \end{array} \tag{28}$$

By Proposition 2.1 $\lambda(\alpha) \in \mathbb{Z}_{\geq 0}$ in all cases.

For $\natural = \natural_{q\mathcal{E}}$ [AMS2, (91)] says that

$$\mathbb{C}[\mathfrak{R}_{q\mathcal{E}}, \natural_{q\mathcal{E}}] \cong \mathrm{End}^+_{\mathcal{D}\mathrm{Lie}(G)_{\mathrm{RS}}}\big((\mathrm{pr}_1)_! \dot{q\mathcal{E}}\big).$$

We denote the algebra constructed in Proposition 2.1, with these extra data, by $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$. Since it is built from an affine Hecke algebra $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{\mathbf{z}})$ and a twisted group algebra $\mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$, we refer to it as a twisted affine Hecke algebra. When $d = 1$ we simply write $\mathcal{H}(G, M, q\mathcal{E})$. We record that

$$\mathcal{H}(G, M, q\mathcal{E}) = \mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})/(\{\mathbf{z}_i - \mathbf{z}_j : 1 \leq i, j \leq d\}). \tag{29}$$

The same argument as for [AMS2, Lemma 2.8] shows that

$$\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}) = \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathrm{End}^+_{\mathcal{D}\mathfrak{g}_{\mathrm{RS}}}\big((\mathrm{pr}_1)_! \dot{q\mathcal{E}}\big). \tag{30}$$

If we are in of the cases (7), then with this interpretation $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ depends canonically on $(G, M, q\mathcal{E})$. In general the algebra $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ is not entirely canonical, since it involves the choice of a decomposition (8).

Lemma 2.3. *$\mathcal{O}(T \times (\mathbb{C}^\times)^d)^{W_{q\mathcal{E}}} = \mathcal{O}(T)^{W_{q\mathcal{E}}} \otimes \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}]$ is a central subalgebra of $\mathcal{H}(G, M, q_\mathcal{E})$. It equals $Z(\mathcal{H}(G, M, q\mathcal{E})$ if $W_{q\mathcal{E}}$ acts faithfully on T.*

Proof. The case $W_{q\mathcal{E}} = 1, d = 1$ is [Lus3, Proposition 3.11]. The general case from readily from that, as observed in [Sol3, §1.2]. □

2.1. Reduction to real central character.

Let $T = T_{\mathrm{un}} \times T_{\mathrm{rs}}$ be the polar decomposition of the complex torus T, in a unitary and a real split part:

$$\begin{aligned} T_{\mathrm{un}} &= \mathrm{Hom}(X^*(T), S^1) = \exp(i\mathfrak{t}_\mathbb{R}), \\ T_{\mathrm{rs}} &= \mathrm{Hom}(X^*(T), \mathbb{R}_{>0}) = \exp(\mathfrak{t}_\mathbb{R}). \end{aligned} \tag{31}$$

Let $t = (t|t|^{-1})\,|t| \in T_{\mathrm{un}} \times T_{\mathrm{rs}}$ denote the polar decomposition of an arbitrary element $t \in T$.

By Lemma 2.3 every irreducible representation of $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ admits a $\mathcal{O}(T \times (\mathbb{C}^\times)^d)^{W_{q\mathcal{E}}}$-character, an element of $T/W_{q\mathcal{E}} \times (\mathbb{C}^\times)^d$. We will refer to this as the central character. Following [BaMo, Definition 2.2] we say that a central character $(W_\mathcal{E} t, \vec{z})$ is "real" if $\vec{z} \in \mathbb{R}^d_{>0}$ and the unitary part $t|t|^{-1}$ is fixed by $W^\circ_\mathcal{E}$.

For $t \in T$ we define $\tilde{Z}_G(t)$ to be the subgroup of G generated by $Z_G(t)$ and the root subgroups for $\alpha \in R(G^\circ, T)$ with $2\alpha \in R(Z_G(t)^\circ, T)$. The group $\tilde{Z}_G(t)$ is such that $R(\tilde{Z}_G(t)^\circ, T)$ consists of the roots $\alpha \in R(G^\circ, T)$ with $s_\alpha(t) = t$. The analogue of $\mathfrak{R}_{q\mathcal{E}}$ for $\tilde{Z}_G(t)$ is $\mathfrak{R}_{q\mathcal{E},t}$, the stabilizer of $R(\tilde{Z}_G(t)^\circ, T) \cap R(P, T)$ in $W_{q\mathcal{E},t}$.

Our first reduction theorem will relate modules of $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ and of $\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})$. Assuming that every $\mathbf{z}_j$ acts via a positive real number, we end up with representations admitting a real central character. To describe the effect on $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weights, we need some preparations. Consider the set

$$W^{\circ,t}_\mathcal{E} = \big\{ w \in W^\circ_\mathcal{E} : w\big(R(\tilde{Z}_G(t)^\circ, T) \cap R(P, T)\big) \subset R(P, T) \big\}.$$

Recall that the parabolic subgroup $P \subset G^\circ$ determines a set of simple reflections and a length function on the Weyl group $W^\circ_\mathcal{E}$.

Lemma 2.4. (*a*) *$W^{\circ,t}_\mathcal{E}$ is the unique set of shortest length representatives of $W^\circ_\mathcal{E}/W(\tilde{Z}_G(t)^\circ, T)$ in $W^\circ_\mathcal{E}$.*

(*b*) $\bigcup_{w \in W_{\mathcal E}^{\circ,t}} w^{-1} \mathfrak t_{\mathbb R}^+$ *equals* $\mathfrak t_{\mathbb R}^{+,t}$*, the analogue of* $\mathfrak t_{\mathbb R}^+$ *for the group* $\tilde Z_G (t)^\circ$*. The same holds for* $\mathfrak t_{\mathbb R}^{*,+}$*.*

(*c*) $\{ x \in \mathfrak t_{\mathbb R} : W_{\mathcal E}^{\circ,t} x \subset \mathfrak t_{\mathbb R}^- \}$ *equals* $\mathfrak t_{\mathbb R}^{-,t}$*, the analogue of* $\mathfrak t_{\mathbb R}^-$ *for* $\tilde Z_G (t)^\circ$*.*

Proof. (a) This is well-known when $\tilde Z_G (t)^\circ$ is a parabolic subgroup of G°, see for example [Hum, 1.10]. The same argument works in the present situation.

(b) Suppose that $x \in \mathfrak t_{\mathbb R}^+$ and $\alpha \in R(\tilde Z_G (t)^\circ, T) \cap R(P,T)$. For all $w \in W_{\mathcal E}^{\circ,t}$ we have $w \alpha \in R(P,T)$, so

$$\langle \alpha \, , \, w^{-1} x \rangle = \langle w \alpha \, , \, x \rangle \geq 0 .$$

Hence $\bigcup_{w \in W_{\mathcal E}^{\circ,t}} w^{-1} \mathfrak t_{\mathbb R}^+ \subset \mathfrak t_{\mathbb R}^{+,t}$. Let S be a sphere in $\mathfrak t_{\mathbb R}$ centred in 0. Then

$$\mathrm{vol}(S) / \mathrm{vol}(S \cap \mathfrak t_{\mathbb R}^+) = |W_{\mathcal E}^\circ| \quad \text{and} \quad \mathrm{vol}(S) / \mathrm{vol}(S \cap \mathfrak t_{\mathbb R}^{+,t}) = |W(\tilde Z_G (t)^\circ, T)| .$$

With part (a) it follows that

$$|W_{\mathcal E}^{\circ,t}| \mathrm{vol}(S \cap \mathfrak t_{\mathbb R}^+) = |W_{\mathcal E}^\circ| \mathrm{vol}(S \cap \mathfrak t_{\mathbb R}^+) / |W(\tilde Z_G (t)^\circ, T)| = \mathrm{vol}(S \cap \mathfrak t_{\mathbb R}^{+,t}). \tag{32}$$

Since $\mathfrak t_{\mathbb R}^+$ is a Weyl chamber for $W_{\mathcal E}^\circ$, the translates $w \mathfrak t_{\mathbb R}^+$ intersect $\mathfrak t_{\mathbb R}^+$ only in a set of measure zero. Hence the left hand side of (32) is the volume of $S \cap \bigcup_{w \in W_{\mathcal E}^{\circ,t}} w^{-1} \mathfrak t_{\mathbb R}^+$. As $\bigcup_{w \in W_{\mathcal E}^{\circ,t}} w^{-1} \mathfrak t_{\mathbb R}^+ \subset \mathfrak t_{\mathbb R}^{+,t}$ and both are cones defined by linear equations coming from roots, the equality (32) shows that they coincide.

The same reasoning applies to $\mathfrak t_{\mathbb R}^*$ and the dual root systems.

(c) The definition of $W_{\mathcal E}^{\circ,t}$ entails $W_{\mathcal E}^{\circ,t} \mathfrak t_{\mathbb R}^{-,t} \subset \mathfrak t_{\mathbb R}^-$. Conversely, suppose that $x \in \mathfrak t_{\mathbb R}$ and that $W_{\mathcal E}^{\circ,t} x \subset \mathfrak t_{\mathbb R}^-$. For every $w \in W_{\mathcal E}^{\circ,t}$ and every $\lambda \in \mathfrak t_{\mathbb R}^{*,+}$:

$$\langle x \, , \, w^{-1} \lambda \rangle = \langle w x \, , \, \lambda \rangle \leq 0 .$$

In view of part (b) for $\mathfrak t_{\mathbb R}^{*,+}$, this means that $x \in \mathfrak t_{\mathbb R}^{-,t}$. $\Box$

Theorem 2.5. *Let* $t \in T_{\mathrm{un}}$*.*

(*a*) *There is a canonical equivalence between the following categories:*
- *finite dimensional* $\mathcal H (\tilde Z_G (t), M, q\mathcal E, \vec{\mathbf z})$*-modules with* $\mathcal O (T \times (\mathbb C^\times)^d)$*-weights in* $t T_{\mathrm{rs}} \times \mathbb R_{>0}^d$*;*
- *finite dimensional* $\mathcal H (G, M, q\mathcal E, \vec{\mathbf z})$*-modules with* $\mathcal O (T \times (\mathbb C^\times)^d)$*-weights in* $W_{q\mathcal E} t T_{\mathrm{rs}} \times \mathbb R_{>0}^d$*.*

It is given by localization of the centre and induction, and we denote it (suggestively) by $\mathrm{ind}_{\mathcal H (\tilde Z_G (t), M, q\mathcal E, \vec{\mathbf z})}^{\mathcal H (G, M, q\mathcal E, \vec{\mathbf z})}$*.*

(*b*) *The above equivalences are compatible with parabolic induction, in the following sense. Let* $Q \subset G$ *be an algebraic subgroup such that* $Q \cap G^\circ$ *is a Levi subgroup of* G° *and* $Q \supset M$*. Then*

$$\mathrm{ind}_{\mathcal H (\tilde Z_G (t), M, q\mathcal E, \vec{\mathbf z})}^{\mathcal H (G, M, q\mathcal E, \vec{\mathbf z})} \circ \mathrm{ind}_{\mathcal H (\tilde Z_Q (t), M, q\mathcal E, \vec{\mathbf z})}^{\mathcal H (\tilde Z_G (t), M, q\mathcal E, \vec{\mathbf z})} = \mathrm{ind}_{\mathcal H (Q, M, q\mathcal E, \vec{\mathbf z})}^{\mathcal H (G, M, q\mathcal E, \vec{\mathbf z})} \circ \mathrm{ind}_{\mathcal H (\tilde Z_Q (t), M, q\mathcal E, \vec{\mathbf z})}^{\mathcal H (Q, M, q\mathcal E, \vec{\mathbf z})} .$$

(*c*) *The set of* $\mathcal O (T \times (\mathbb C^\times)^d)$*-weights of* $\mathrm{ind}_{\mathcal H (\tilde Z_G (t), M, q\mathcal E, \vec{\mathbf z})}^{\mathcal H (G, M, q\mathcal E, \vec{\mathbf z})} (V)$ *is*

$$\big\{ (w x, \vec z) : w \in \mathfrak R_{q\mathcal E} W_{\mathcal E}^{\circ,t}, (x, \vec z) \text{ is a } \mathcal O (T \times (\mathbb C^\times)^d)\text{-weight of } V \big\} .$$

Proof. (a) The case $d = 1, \mathfrak R_{q\mathcal E} = 1$ was proven in [Lus3, Theorem 8.6].

Let $\mathfrak{R}^+_{q\mathcal{E}} \to \mathfrak{R}_{q\mathcal{E}}$ be a central extension as in (27). Then

$$\begin{aligned} \mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}) &= \mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes p_\natural \mathbb{C}[\mathfrak{R}^+_{q\mathcal{E}}], \\ \mathcal{H}(\tilde{Z_G}(t), M, q\mathcal{E}, \vec{\mathbf{z}}) &= \mathcal{H}(\tilde{Z_{G^\circ}}(t), M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes p_\natural \mathbb{C}[\mathfrak{R}^+_{q\mathcal{E}}]. \end{aligned} \tag{33}$$

As $p_\natural \in \mathbb{C}[\ker(\mathfrak{R}^+_{q\mathcal{E}} \to \mathfrak{R}_{q\mathcal{E}})]$ is a central idempotent, we may just as well establish the analogous result for the algebras

$$\mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}} \quad \text{and} \quad \mathcal{H}(\tilde{Z_{G^\circ}}(t), M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}}.$$

Since we are dealing with finite dimensional representations only, we can decompose them according to the (generalized) weights for the action of the centre. Fix $(x, \vec{z}) \in T_{\mathrm{rs}} \times \mathbb{R}^d_{>0}$. Denote the category of finite dimensional A-modules with weights in U by $\mathrm{Mod}_{f,U}(A)$. We compare the categories

$$\begin{aligned} &\mathrm{Mod}_{f, W_{q\mathcal{E},t} tx \times \{\vec{z}\}} \big(\mathcal{H}(\tilde{Z_{G^\circ}}(t), M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}} \big), \\ &\mathrm{Mod}_{f, W_{q\mathcal{E},t} tx \times \{\vec{z}\}} \big(\mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}} \big). \end{aligned} \tag{34}$$

The most appropriate technique to handle the general case is analytic localization, as in [Opd1, §4] (but there with fixed parameters $z_1, \ldots, z_d$). For a submanifold $U \subset T \times (\mathbb{C}^\times)^d$, let $C^{an}(U)$ be the algebra of complex analytic functions on U. We assume that U is $W_{q\mathcal{E}}$-stable and Zariski-dense. Then the restriction map $\mathcal{O}(T \times (\mathbb{C}^\times)^d) \to C^{an}(U)$ is injective, and we can form the algebra

$$\mathcal{H}^{an}(U) := C^{an}(U)^{W_{q\mathcal{E}}} \underset{\mathcal{O}(T \times (\mathbb{C}^\times)^d)^{W_{q\mathcal{E}}}}{\otimes} \mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}}. \tag{35}$$

As observed in [Opd1, Proposition 4.3], the finite dimensional modules of $\mathcal{H}^{an}(U)$ can be identified with the finite dimensional modules of $\mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}}$ with $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weights in U.

In [Sol3, Conditions 2.1] it is described how one can find a sufficiently small open neighborhood $U_0 \subset T \times (\mathbb{C}^\times)^d$ of $(x, \vec{z})$. We take $U = W_{q\mathcal{E}} U_0$ and $\tilde{U} = W_{q\mathcal{E},t} U_0$. By Lusztig's first reduction theorem, in the version [Sol3, Theorem 2.1.2], there is a natural inclusion of

$$\mathcal{H}^{an}_t(\tilde{U}) := C^{an}(\tilde{U})^{W_{q\mathcal{E},t}} \underset{\mathcal{O}(T \times (\mathbb{C}^\times)^d)^{W_{q\mathcal{E},t}}}{\otimes} \mathcal{H}(\tilde{Z_{G^\circ}}(t), M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}}$$

in $\mathcal{H}^{an}(U)$, which moreover is a Morita equivalence. Hence the composed functor

$$\begin{aligned} \mathrm{ind}_{\mathcal{H}(\tilde{Z_{G^\circ}}(t), M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}}}^{\mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}}} &: \mathrm{Mod}_{f,\tilde{U}} \big(\mathcal{H}(\tilde{Z_{G^\circ}}(t), M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}} \big) \to \\ &\mathrm{Mod}_f \big(\mathcal{H}^{an}_t(\tilde{U}) \big) \to \mathrm{Mod}_f \big(\mathcal{H}^{an}(U) \big) \to \mathrm{Mod}_{f,U} \big(\mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}} \big) \end{aligned}$$

is an equivalence of categories. We specialize this at $W_{q\mathcal{E},t} tx \times \{\vec{z}\} \subset U$ and we restrict to modules on which $p_\natural$ acts as the identity. Via (34) and (33) this gives the required equivalence of categories $\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t), M, q\mathcal{E}, \vec{\mathbf{z}})}^{\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})}$.

(b) We just showed that the above functor is really induction between localizations of the indicated algebras. Similar remarks apply to the functor $\mathrm{ind}_{\mathcal{H}(Q, M, q\mathcal{E}, \vec{\mathbf{z}})}^{\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})}$. Thus the acclaimed compatibility with parabolic induction is just an instance of the transitivity of induction.

(c) Lemma 2.4.a and the constructions in [Sol3, §2.1] entail that

$$\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V) \cong \mathbb{C}[\mathfrak{R}_{q\mathcal{E}} W_{\mathcal{E}}^{\circ,t}, \natural_{q\mathcal{E}}] \underset{\mathbb{C}[\mathfrak{R}_{q\mathcal{E},t},\natural_{q\mathcal{E}}]}{\otimes} V \tag{36}$$

as $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-modules. Notice that the group $\mathfrak{R}_{q\mathcal{E},t}$ acts from the right on $\mathfrak{R}_{q\mathcal{E}} W_{\mathcal{E}}^{\circ,t}$, because it stabilizes $R(\tilde{Z}_G(t)^\circ, T) \cap R(P,T)$. Since

$$\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}}) \cong \mathcal{H}(\tilde{Z}_G(t)^\circ M, M, q\mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\mathfrak{R}_{q\mathcal{E},t}, \natural_{q\mathcal{E}}],$$

the $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weights of V come in full $\mathfrak{R}_{q\mathcal{E},t}$-orbits. It was observed in the proof of [Opd1, Proposition 4.20] that the $\mathcal{O}(T \times \mathbb{C}^\times)^d)$-weights of $\mathbb{C}w \otimes V$ $(w \in W_{\mathcal{E}}^{\circ,t})$ are precisely $(wx, \vec{z})$ with $(x, \vec{z})$ a $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weight of V. Multiplication by N_γ $(\gamma \in \mathfrak{R}_{q\mathcal{E}})$ just changes a weight $(x, \vec{z})$ to $(\gamma x, \vec{z})$. These observations and (36) prove that the $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weights of $\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ are as stated. $\square$

In our reduction process we would like to preserve the analytic properties from [AMS2, §3.5]. Just as in [AMS2, (79)], we can define $\mathcal{O}(T)$-weights for modules of affine Hecke algebras or extended versions such as $\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})$. We denote the set of $\mathcal{O}(T)$-weights of a module V for such an algebra by $\mathrm{Wt}(V)$. We can apply the polar decomposition (31) to it, which gives a set $|\mathrm{Wt}(V)| \subset T_{\mathrm{rs}}$.

Let us recall the definitions of temperedness and discrete series from [Opd1, §2.7].

Definition 2.6. Let V be a finite dimensional $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$-module. We say that V is tempered (respectively anti-tempered) if $|\mathrm{Wt}(V)| \subset \exp(\mathfrak{t}_{\mathbb{R}}^-)$, respectively $\subset \exp(-\mathfrak{t}_{\mathbb{R}}^-)$.

We call V discrete series (resp. anti-discrete series) if $|\mathrm{Wt}(V)| \subset \exp(\mathfrak{t}_{\mathbb{R}}^{--})$, respectively $\subset \exp(-\mathfrak{t}_{\mathbb{R}}^{--})$.

The module V is essentially discrete series if its restriction to $\mathcal{H}(G^\circ_{\mathrm{der}} M, M, q\mathcal{E}, \vec{\mathbf{z}})$ is discrete series, or equivalently if $|\mathrm{Wt}(V)| \subset \exp(Z(\mathfrak{g}) \oplus \mathfrak{t}_{\mathbb{R}}^{--})$.

The next result fills a gap in [Sol3, Theorem 2.3.1], where it was used between the lines. Similar results, for G°_{der} only and with somewhat different notions of temperedness and discrete series, were proven in [Lus5, Lemmas 3.4 and 3.5].

Proposition 2.7. *The equivalence from Theorem 2.5.a, and its inverse preserve:*
(*a*) *(anti-)temperedness,*
(*b*) *the discrete series,*
(*c*) *the essentially discrete series property, provided that* $R(\tilde{Z}_G(t)^\circ, T)$ *has full rank in* $R(G^\circ, T)$.

Remark. The extra condition for essentially discrete series representations is necessary, for the centre of $\tilde{Z}_G(t)^\circ$ can be of higher dimension than that of G°.

Proof. Let V be a finite dimensional $\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})$-module with $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weights in $tT_{\mathrm{rs}} \times \mathbb{R}^d_{>0}$.

(a) The $\mathcal{O}(T)$-weights of $\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ were given in Theorem 2.5.c. As $\log = \exp^{-1} : T_{\mathrm{rs}} \to \mathfrak{t}_{\mathbb{R}}$ is $W_{q\mathcal{E}}$-equivariant, it entails that

$$\log \big| \mathrm{Wt}\big(\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} V \big) \big| = \mathfrak{R}_{q\mathcal{E}} W_{\mathcal{E}}^{\circ,t} \log |\mathrm{Wt}(V)|.$$

Recall from Lemma 2.4.c that

$$\mathfrak{t}_{\mathbb{R}}^{-,t} = \{x \in \mathfrak{t}_{\mathbb{R}} : W_{\mathcal{E}}^{\circ,t} x \subset \mathfrak{t}_{\mathbb{R}}^{-}\} = \{x \in \mathfrak{t}_{\mathbb{R}} : \mathfrak{R}_{q\mathcal{E}} W_{\mathcal{E}}^{\circ,t} x \subset \mathfrak{t}_{\mathbb{R}}^{-}\}.$$

Comparing these with the definition of (anti-)temperedness for G and for $\tilde{Z}_G(t)$, we see that V is (anti-)tempered if and only if $\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ is so.
(b) We have to assume that $Z(G^\circ)$ is finite, for otherwise $\exp(\mathfrak{t}_{\mathbb{R}}^{--})$ is empty and there are no discrete series representations on any side of the equivalences.

Suppose that V is discrete series. Then $\tilde{Z}_G(t)^\circ$ is semisimple, so $R(\tilde{Z}_G(t)^\circ, T)$ is of full rank in $R(G^\circ, T)$. This implies that $\mathfrak{t}_{\mathbb{R}}^{--,t}$ is an open subset of $\mathfrak{t}_{\mathbb{R}}^{--}$. The same argument as for part (a) shows that $\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ is discrete series.

Conversely, suppose that $\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ is discrete series. It is tempered, so V is tempered and $|\mathrm{Wt}(V)| \subset \exp(\mathfrak{t}_{\mathbb{R}}^{-,t})$. Suppose that $\tilde{Z}_G(t)^\circ$ is not semisimple. Then

$$\mathfrak{t}_Z := \mathrm{Lie}\big(Z(\tilde{Z}_G(t)^\circ)\big) = \bigcap\nolimits_{\alpha \in R(\tilde{Z}_G(t)^\circ, T)} \ker \alpha$$

has positive dimension. In particular $\mathfrak{t}_Z^*$ contains nonzero elements $\lambda \in \mathfrak{t}_{\mathbb{R}}^{*,+}$, for example the sum of the fundamental weights for simple roots not in $\mathbb{R}R(\tilde{Z}_G(t)^\circ, T)$. Let $t' \in T$ be any weight of V. Then $\log|t'| \in \mathfrak{t}_{\mathbb{R}}^{-,t} \subset \mathrm{Lie}(\tilde{Z}_G(t)^\circ_{\mathrm{der}})$. Hence $\langle \log|t'| \,,\, \lambda\rangle = 0$, which means that $\log|t'| \in \mathfrak{t}_{\mathbb{R}}^{-} \setminus \mathfrak{t}_{\mathbb{R}}^{--}$. But t' is also a weight of $\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E})}^{\mathcal{H}(G,M,q\mathcal{E})}(V)$, and that is a discrete series representation, so $\log|t'| \in \mathfrak{t}_{\mathbb{R}}^{--}$. This contractiction shows that $\tilde{Z}_G(t)^\circ$ is semisimple.

Suppose now that $\log|t'|$ does not lie in the interior of $\mathfrak{t}_{\mathbb{R}}^{-,t}$. Then it is orthogonal to a nonzero element λ' in the boundary of $\mathfrak{t}_{\mathbb{R}}^{*,+,t}$. By Lemma 2.4.b we can choose a $w \in W_{\mathcal{E}}^{\circ,t}$ such that $w\lambda' \in \mathfrak{t}_{\mathbb{R}}^{*,+}$. Theorem 2.5.c wt' is a weight of $\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$, and it satisfies

$$\langle \log|wt'| \,,\, w\lambda'\rangle = \langle \log|t'| \,,\, \lambda'\rangle = 0.$$

This shows that $\log|wt'| \notin \mathfrak{t}_{\mathbb{R}}^{--}$, which contradicts that $\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ is discrete series. Therefore $\log|t'|$ belongs to $\mathfrak{t}_{\mathbb{R}}^{--,t}$. As t' was an arbitrary weight of V, this proves that V is discrete series.
(c) Suppose that $\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ is essentially discrete series. Its restriction to $\mathcal{H}(G^\circ_{\mathrm{der}}M, M, q\mathcal{E}, \vec{\mathbf{z}})$ is discrete series, so by what we have just proven V is discrete series as a $\mathcal{H}(\widetilde{Z_{G^\circ_{\mathrm{der}}M}}(t), M, q\mathcal{E}, \vec{\mathbf{z}})$-module, and $\widetilde{Z_{G^\circ_{\mathrm{der}}}}(t)^\circ$ is semisimple. Then the restriction of V to the smaller algebra $\mathcal{H}(\tilde{Z}_G(t)^\circ_{\mathrm{der}}M, M, q\mathcal{E}, \vec{\mathbf{z}})$ is also discrete series, so V is essentially discrete series.

Conversely, suppose that V is essentially discrete series. The assumption that $R(\tilde{Z}_G(t)^\circ, T)$ has full rank in $R(G^\circ, T)$ implies that $Z(G^\circ)$ is also the centre of $\tilde{Z}_G(t)^\circ$. The same argument as in the tempered and the discrete series case shows that

$$\big|\mathrm{Wt}\big(\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} V\big)\big| \subset \exp(\mathfrak{t}_{\mathbb{R}}^{--} \oplus Z(\mathfrak{g})).$$

This means that $\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ is essentially discrete series. $\square$

Suppose that $t' \in W_{q\mathcal{E}} t$. Then we can apply Theorem 2.5.a also with t' instead of t, and that should give essentially the same equivalence of categories. We check this in a slightly more general setting, which covers all $t' \in T \cap \mathrm{Ad}(G)t$. We note that

$$T \cap \mathrm{Ad}(G)t = T \cap \mathrm{Ad}(N_G(T))t \supset W_{q\mathcal{E}} t.$$

Let $g \in N_G(M) = N_G(T)$, with image $\bar{g}$ in $N_G(M)/M$. Conjugation with g yields an algebra isomorphism

$$\begin{aligned} &\mathrm{Ad}(g) : \mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}}) \to \mathcal{H}(\tilde{Z}_G(gtg^{-1}), M, \mathrm{Ad}(g^{-1})^* q\mathcal{E}, \vec{\mathbf{z}}), \\ &\mathrm{Ad}(N_w) = N_{\bar{g}w\bar{g}^{-1}}, \quad \mathrm{Ad}(g)\theta_x = \theta_{x \circ \mathrm{Ad}(g^{-1})} = \theta_{\bar{g}x}, \quad \mathrm{Ad}(g)\mathbf{z}_j = \mathbf{z}_j, \end{aligned} \tag{37}$$

where $w \in W_{q\mathcal{E}}$ and $x \in X^*(T)$. Notice that this depends only on g through its class in $N_G(M)/M$.

Lemma 2.8. *Let $t \in T_{\mathrm{un}}$ and $g \in N_G(M)$. Then*

$$\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} = \mathrm{Ad}(g)^* \circ \mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(gtg^{-1}),M,\mathrm{Ad}(g^{-1})^* q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,\mathrm{Ad}(g^{-1})^* q\mathcal{E},\vec{\mathbf{z}})} \circ \mathrm{Ad}(g^{-1})^*$$

as functors between the appropriate categories of modules of these algebras (as specified in Theorem 2.5).

Remark. This result was used, but not proven, in [Lus4, §4.9 and §5.20] and [Sol3, Theorem 2.3.1].

Proof. Our argument for Theorem 2.5.a, with (27), shows how several relevant results can be extended from $\mathcal{H}(G^\circ M, M, q\mathcal{E}, \vec{\mathbf{z}})$ to $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$. This justifies that below we use some results from [Lus3], which were formulated only for $\mathcal{H}(G^\circ M, M, q\mathcal{E})$.

Let (π, V) be a finite dimensional $\mathcal{H}(G, M, q\mathcal{E})$-module with $\mathcal{O}(T \times \mathbb{C}^\times)$-weights in $W_{q\mathcal{E}} t T_{\mathrm{rs}} \times \mathbb{R}_{>0}$. In [Lus3, §8] V is decomposed canonically as $\bigoplus_{t' \in W_{q\mathcal{E}} t} V_{t' T_{\mathrm{rs}}}$, where $V_{t' T_{\mathrm{rs}}}$ is the sum of all generalized $\mathcal{O}(T)$-weight spaces with weights in $t' T_{\mathrm{rs}}$. Then $V_{t' T_{\mathrm{rs}}}$ is a module for $\mathcal{H}(\tilde{Z}_G(t'), M, q\mathcal{E})$ and

$$V = \mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t'),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} (V_{t' T_{\mathrm{rs}}}). \tag{38}$$

Assume that $g \in N_G(M, q\mathcal{E})$, so $\bar{g} \in W_{q\mathcal{E}}$. Then $V_{t T_{\mathrm{rs}}}$ and $V_{gtg^{-1} T_{\mathrm{rs}}}$ are related via multiplication with an element $\tau_{\bar{g}}$, which lives in a suitable localization of $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ [Lus3, §5]. We can rewrite the right hand side of (38) as

$$\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} \big(\tau_{\bar{g}} V_{g^{-1}tg T_{\mathrm{rs}}} \big) = \tau_{\bar{g}} \big(\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(gtg^{-1}),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} (V_{g^{-1}tg T_{\mathrm{rs}}}) \big). \tag{39}$$

From [Lus3, §8.8] and [Sol1, Lemma 4.2] we see that the effect of conjugation by $\tau_{\bar{g}}$ on $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ and $\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})$ boils down to the algebra isomorphism (37). The right hand side of (39) becomes

$$\mathrm{Ad}(g)^* \circ \mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(gtg^{-1}),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}}} \circ \mathrm{Ad}(g^{-1})^* \big(V_{t T_{\mathrm{rs}}} \big),$$

which proves the lemma for such g.

Now we consider a general $g \in N_G(M)$. We will analyse

$$\mathrm{Ad}(g)^* \circ \mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(gtg^{-1}),M,\mathrm{Ad}(g^{-1})^* q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,\mathrm{Ad}(g^{-1})^* q\mathcal{E},\vec{\mathbf{z}})} \circ \mathrm{Ad}(g^{-1})^* \big(V_{t T_{\mathrm{rs}}} \big). \tag{40}$$

From the above we see that the underlying vector space is

$$\bigoplus_{w \in g W_{q\mathcal{E}} g^{-1} / g W_{q\mathcal{E},t} g^{-1}} \tau_w \big(\mathrm{Ad}(g^{-1})^* V_{t T_{\mathrm{rs}}} \big) = \bigoplus_{w \in W_{q\mathcal{E}} / W_{q\mathcal{E},t}} \mathrm{Ad}(g^{-1})^* \tau_w V_{t T_{\mathrm{rs}}} = \mathrm{Ad}(g^{-1})^* V.$$

The action of $\mathcal{H}(\tilde{Z}_G(gtg^{-1}), M, \mathrm{Ad}(g^{-1})^* q\mathcal{E}, \vec{\mathbf{z}}) = \mathrm{Ad}(g)\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})$ works out to

$$(\mathrm{Ad}(g)h) \cdot (\mathrm{Ad}(g^{-1})^* v) = \mathrm{Ad}(g^{-1})^*(h \cdot v).$$

Thus (40) can be identified with V. □

2.2. **Parametrization of irreducible representations.**

Next we want to reduce from $\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})$-modules to modules over $\mathbb{H}(G_t, M, q\mathcal{E}, \vec{\mathbf{r}})$. The exponential map for $T \times \mathbb{C}^\times$ gives a $W_{q\mathcal{E},t}$-equivariant map

$$\exp_t : \mathfrak{t} \oplus \mathbb{C}^d \to T \times (\mathbb{C}^\times)^d, \qquad \exp_t(x, r_1, \ldots, r_d) = (t \exp(x), \exp r_1, \ldots, \exp r_d).$$

Notice that the restriction $\exp_t : \mathfrak{t}_\mathbb{R} \oplus \mathbb{R}^d \to tT_{\mathrm{rs}} \times \mathbb{R}^d_{>0}$ is a diffeomorphism.

Theorem 2.9. *Let* $t \in T_{\mathrm{un}}$.

(*a*) *There is a canonical equivalence between the following categories:*
- *finite dimensional* $\mathbb{H}(G_t, M, q\mathcal{E}, \vec{\mathbf{r}})$*-modules with* $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C}^d)$*-weights in* $\mathfrak{t}_\mathbb{R} \oplus \mathbb{R}^d$*;*
- *finite dimensional* $\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})$*-modules with* $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$*-weights in* $tT_{\mathrm{rs}} \times \mathbb{R}^d_{>0}$.

It is given by localization with respect to central ideals in combination with the map $\exp_t$*. We denote this equivalence by* $(\exp_t)_*$.

(*b*) *The functor* $(\exp_t)_*$ *is compatible with parabolic induction, in the following sense. Let* $Q \subset G$ *be an algebraic subgroup such that* $Q \cap G^\circ$ *is a Levi subgroup of* G° *and* $Q \supset M$*. Then*

$$\mathrm{ind}_{\mathcal{H}(\tilde{Z}_Q(t), M, q\mathcal{E}, \vec{\mathbf{z}})}^{\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})} \circ (\exp_t^Q)_* = (\exp_t)_* \circ \mathrm{ind}_{\mathbb{H}(Q_t, M, q\mathcal{E}, \vec{\mathbf{z}})}^{\mathbb{H}(G_t, M, q\mathcal{E}, \vec{\mathbf{z}})}.$$

(*c*) *The functor* $(\exp_t)_*$ *preserves the underlying vector space of a representation, and it transforms a* $S(\mathfrak{t}^* \oplus \mathbb{C}^d)$*-weight* $(x, \vec{r})$ *into a* $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$*-weight* $\exp_t(x, \vec{r})$.

(*d*) *The functors* $(\exp_t)_*$ *and* $(\exp_t)_*^{-1}$ *preserve (anti-)temperedness and (essentially) discrete series.*

Proof. (a) The case $d = 1, \mathfrak{R}_{q\mathcal{E}} = 1$ was proven in [Lus3, Theorem 9.3].

For the general case we use the similar techniques and notations as in the proof of Theorem 2.5.a. By the same argument as over there, it suffices to compare the categories

$$\begin{aligned} &\mathrm{Mod}_{f, W_{q\mathcal{E},t} tx \times \{\vec{z}\}} \big(\mathcal{H}(\tilde{Z}_G(t)^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t} \big), \\ &\mathrm{Mod}_{f, W_{q\mathcal{E},t} \log(x) \times \{\log(\vec{z})\}} \big(\mathbb{H}(Z_G(t)^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t} \big). \end{aligned} \tag{41}$$

Recall from (28) that the parameter functions for these algebras are related by

$$\begin{array}{lcll} c_t(\alpha) & = & 2\lambda(\alpha) & 2\alpha \notin R(\tilde{Z}_G(t)^\circ, T), \\ c_t(\alpha) & = & \lambda(\alpha) + \lambda^*(\alpha) & 2\alpha \in R(\tilde{Z}_G(t)^\circ, T), \alpha(t) = 1, \\ c_t(2\alpha)/2 & = & \lambda(\alpha) - \lambda^*(\alpha) & 2\alpha \in R(\tilde{Z}_G(t)^\circ, T), \alpha(t) = -1. \end{array} \tag{42}$$

Let us define $k : R(\tilde{Z}_G(t)^\circ, T)_{\mathrm{red}} \to \mathbb{R}$ by

$$\begin{array}{lcll} k(\alpha) & = & 2\lambda(\alpha) & 2\alpha \notin R(\tilde{Z}_G(t)^\circ, T), \\ k(\alpha) & = & \lambda(\alpha) + \alpha(t)\lambda^*(\alpha) & 2\alpha \in R(\tilde{Z}_G(t)^\circ, T). \end{array} \tag{43}$$

The only difference between $\mathbb{H}(\mathfrak{t}, W(\tilde{Z_G}(t)^\circ, T), k\vec{\mathbf{r}})$ and $\mathbb{H}(Z_G(t)^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$ arises from roots $\alpha \in R(\tilde{Z_G}(t)^\circ, T) \setminus R(Z_G(t)^\circ, T)$ with $\alpha(t) = -1$. The corresponding braid relations are

$$\begin{array}{llll} N_{s_\alpha}\xi - {}^{s_\alpha}\xi N_{s_\alpha} & = & (\lambda(\alpha) - \lambda^*(\alpha))\mathbf{r}_j(\xi - {}^{s_\alpha}\xi)/\alpha & \text{in } \mathbb{H}(\mathfrak{t}, W(\tilde{Z_G}(t)^\circ, T), k\vec{\mathbf{r}}), \\ N_{s_{2\alpha}}\xi - {}^{s_{2\alpha}}\xi N_{s_{2\alpha}} & = & c_t(2\alpha)\mathbf{r}_j(\xi - {}^{s_{2\alpha}}\xi)/(2\alpha) & \text{in } \mathbb{H}(Z_G(t)^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}}). \end{array}$$

Since $s_\alpha = s_{2\alpha}$ and $c_t(2\alpha) = 2(\lambda(\alpha) - \lambda^*(\alpha))$, these two braid relations are equivalent, and we may identify

$$\mathbb{H}(\mathfrak{t}, W(\tilde{Z_G}(t)^\circ, T), k\vec{\mathbf{r}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t} = \mathbb{H}(Z_G(t)^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t}. \tag{44}$$

Let $V \subset \mathfrak{t} \times \mathbb{C}^d$ be a $W_{q\mathcal{E},t}$-stable, Zariski-dense submanifold. Like in (35) we can form the algebra

$$\mathbb{H}^{an}_t(V) := C^{an}(V)^{W_{q\mathcal{E},t}} \underset{\mathcal{O}(\mathfrak{t} \oplus \mathbb{C}^d)^{W_{q\mathcal{E},t}}}{\otimes} \mathbb{H}(\mathfrak{t}, W(\tilde{Z_G}(t)^\circ, T), k\vec{\mathbf{r}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t}.$$

The argument for [Opd1, Proposition 4.3] shows that its finite dimensional modules are precisely the finite dimensional $\mathbb{H}(\mathfrak{t}, W(\tilde{Z_G}(t)^\circ, T), k\vec{\mathbf{r}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t}$-modules with $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C}^d)$-weights in V. If $\exp_t$ is injective on V, it induces an algebra isomorphism

$$\exp_t^* : C^{an}(\exp_t(V))^{W_{q\mathcal{E},t}} \to C^{an}(V)^{W_{q\mathcal{E},t}}. \tag{45}$$

We suppose in addition that V is contained in a sufficiently small open neighborhood of $\mathfrak{t}_\mathbb{R} \oplus \mathbb{R}^d$. In view of the relations between the parameters (42) and (43), we can apply [Sol3, Theorem 2.1.4.b]. It shows that (45) extends to an isomorphism of $C^{an}(V)^{W_{q\mathcal{E},t}}$-algebras

$$\Phi_t : C^{an}(\exp_t(V))^{W_{q\mathcal{E},t}} \underset{\mathcal{O}(T \times (\mathbb{C}^\times)^d)^{W_{q\mathcal{E},t}}}{\otimes} \mathcal{H}(\tilde{Z_G}(t)^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t} \to \mathbb{H}^{an}_t(V),$$

which is the identity on $\mathbb{C}[\mathfrak{R}^+_{q\mathcal{E},t}]$.

Choosing for V a small neighborhood of $W_{q\mathcal{E},t}\log(x) \times \{\log(\vec{z})\}$ in $\mathfrak{t} \oplus \mathbb{C}^d$, Φ_t induces an equivalence between the categories of modules with weights in, respectively, $W_{q\mathcal{E},t}tx \times \{\vec{z}\}$ and $W_{q\mathcal{E},t}\log(x) \times \{\log(\vec{z})\}$. In view of [Opd1, Proposition 4.3] and (44), this provides the equivalence between the categories (41).

Since Φ_t fixes $p_\natural \in \mathbb{C}[\mathfrak{R}^+_{q\mathcal{E},t}]$, we can restrict that equivalence to modules on which $p_\natural$ acts as the identity.

(b) For G° this is shown in [BaMo, Theorem 6.2] and [Sol2, Proposition 6.4]. Extending G° to a disconnected group boils down to extending the involved algebras by $\mathbb{C}[\mathfrak{R}_{q\mathcal{E},t}, \natural_{q\mathcal{E}}]$ or $\mathbb{C}[\mathfrak{R}^Q_{q\mathcal{E},t}, \natural_{q\mathcal{E}}]$. As we noted in proof of part (a), the algebra homomorphism Φ_t used to define $(\exp_t)_*$ is the identity on $\mathbb{C}[\mathfrak{R}_{q\mathcal{E},t}, \natural_{q\mathcal{E}}] \subset \mathbb{C}[\mathfrak{R}^+_{q\mathcal{E},t}]$. Hence this extension works the same on both sides of the equivalence, and the argument given in [Sol2, §6] generalizes to the current setting.

(c) By construction [Sol3, §2.1] $(\exp_t)_*\pi = \pi \circ \exp_t^*$ as $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-representations. (For $f \in \mathcal{O}(T \times (\mathbb{C}^\times)^d)$ the action of $f \circ \exp_t$ on the vector space underlying π is defined via a suitable localization.) This immediately implies that $(\exp_t)_*$ has the effect of $\exp_t$ on weights.

(d) This result generalizes the observations made in [Slo, (2.11)]. Let V be a finite dimensional $\mathcal{H}(\tilde{Z_G}(t), M, q\mathcal{E})$-module with $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weights in $tT_{\mathrm{rs}} \times \mathbb{R}_{>0}$. By part (b)

$$\mathrm{Wt}((\exp_t)_*^{-1}V) = \exp_t^{-1}(\mathrm{Wt}(V)) \subset \mathfrak{t}_\mathbb{R}.$$

By assumption $t \in T_{\mathrm{un}}$, so we get

$$|\mathrm{Wt}(V)| = \exp\big(\Re\big(\mathrm{Wt}((\exp_t)_*^{-1} V)\big)\big).$$

Comparing [AMS2, Definition 3.24] and Definition 2.6, we see that $(\exp_t)_*$ and $(\exp_t)_*^{-1}$ preserve (anti-)temperedness and the discrete series. With [AMS2, Definition 3.27] we see that "essentially discrete series" is also respected. □

Theorems 2.5 and 2.9 together provide an equivalence between $\mathbb{H}(G_t, M, q\mathcal{E}, \vec{\mathbf{r}})$-modules with central character in $\mathfrak{t}_\mathbb{R}/W_{q\mathcal{E},t} \times \mathbb{R}^d$ and $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$-modules with central character in $W_{q\mathcal{E}} t T_{\mathrm{rs}}/W_{q\mathcal{E}} \times \mathbb{R}^d_{>0}$, where $t \in T_{\mathrm{un}}$.

Recall from [AMS2, Corollary 3.23] and Theorem 1.4 that we can parametrize $\mathrm{Irr}_{\vec{r}}(\mathbb{H}(G_t, M, q\mathcal{E}, \vec{\mathbf{r}}))$ with $N_{G_t}(M)/M$-orbits of triples $(\sigma_0, \mathcal{C}, \mathcal{F})$, where $\sigma_0 \in \mathfrak{t}$, $\mathcal{C}$ is a nilpotent $Z_{G_t}(\sigma_0)$-orbit in $Z_\mathfrak{g}(\sigma_0)$ and $\mathcal{F}$ is an irreducible $Z_{G_t}(\sigma_0)$-equivariant local system on $\mathcal{C}$ such that $\Psi_{Z_{G_t}(\sigma_0)}(\mathcal{C}, \mathcal{F}) = (M, \mathcal{C}_v^M, q\mathcal{E})$, up to $Z_{G_t}(\sigma_0)$-conjugacy.

To find all irreducible representations with $S(\mathfrak{t}^*)^{W_{q\mathcal{E}}}$-character in $\mathfrak{t}_\mathbb{R}$, and those are all we need for the relation with affine Hecke algebras, it suffices to consider such triples $(\sigma_0, \mathcal{C}, \mathcal{F})$ with $\sigma_0 \in \mathfrak{t}_\mathbb{R}$. To phrase things more directly in terms of the group G, we allow t to vary in T_{un} and we replace σ_0 by $t' = t \exp(\sigma_0) \in t T_{\mathrm{rs}}$. In other words, we consider triples $(t', \mathcal{C}, \mathcal{F})$ such that:

- $t' \in T$ with unitary part $t = t'|t'|^{-1}$;
- $\mathcal{C}$ is a nilpotent $Z_G(t')$-orbit in $Z_\mathfrak{g}(t') = \mathrm{Lie}(G_{t'})$.
- $\mathcal{F}$ is an irreducible $Z_G(t')$-equivariant local system on $\mathcal{C}$ with $q\Psi_{Z_G(t')}(\mathcal{C}, \mathcal{F}) = (M, \mathcal{C}_v^M, q\mathcal{E})$, up to $Z_G(t')$-conjugacy.

To such a triple we can associate the standard $\mathbb{H}(G_t, M, q\mathcal{E}, \vec{\mathbf{r}})$-modules

$$E_{y, \log|t'| + \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right), \vec{r}, \rho} \quad \text{and} \quad \mathrm{IM}^* E_{y, -\log|t'| + \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right), \vec{r}, \rho}, \tag{46}$$

where $y \in \mathcal{C}$ and ρ is the representation of $\pi_0(Z_G(t', y))$ on $\mathcal{F}_y$. Furthermore $\gamma : \mathrm{SL}_2(\mathbb{C}) \to Z_G(t')^\circ$ is an algebraic homomorphism with

$$\mathrm{d}\gamma\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = y \quad \text{and} \quad \mathrm{d}\gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{t} \tag{47}$$

and $\mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right)$ is given by (14). The modules (46) have distinguished irreducible quotients

$$M_{y, \log|t'| + \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right), \vec{r}, \rho} \quad \text{and} \quad \mathrm{IM}^* M_{y, -\log|t'| + \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right), \vec{r}, \rho}.$$

By [AMS2, Corollary 3.23] all these representations depend only on the $N_{G_t}(M)/M$-orbit of $(t', \mathcal{C}, \mathcal{F})$, not on the additional choices.

For $\vec{z} \in \mathbb{R}^d_{>0}$ we consider the irreducible $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$-module

$$\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t), M, q\mathcal{E}, \vec{\mathbf{z}})}^{\mathcal{H}(G, M, \mathcal{E}, \vec{\mathbf{z}})} (\exp_t)_* \, \mathrm{IM}^* \, M_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{smallmatrix}\right) - \log|t'|, \log \vec{z}, \rho}. \tag{48}$$

Lemma 2.10. *Fix $\vec{z} \in \mathbb{R}^d_{>0}$. The representations* (48) *provide a bijection between $\mathrm{Irr}_{\vec{z}}(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ and $N_G(M)/M$-orbits of triples $(t', \mathcal{C}, \mathcal{F})$ as above.*

Proof. For irreducible $\mathcal{H}(G, M, q\mathcal{E})$-representations with central character in $W_{q\mathcal{E}} t T_{\mathrm{rs}} \times \mathbb{R}_{>0}$ this follows from [AMS2, Corollary 3.23] and Theorems 2.9 and 2.5. We note that at this point we still have to consider $N_{G_t}(M)/M$-conjugacy classes of parameters $(t', \mathcal{C}, \mathcal{F})$.

With Theorem 2.5.a we extend this to the whole of $\mathrm{Irr}_{\vec{z}}(\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}}))$. By Lemma 2.8 the parametrization does not depend on the choice of a unitary element t in a $W_{q\mathcal{E}}$-orbit in T, and the representation (48) depends only on the $N_G(M)/M$-orbit of $(t',\mathcal{C},\mathcal{F})$. $\square$

To simplify the parameters, we would like to get rid of the restriction $t' \in T$ – we would rather allow any semisimple element of G°. It is also convenient to replace $\mathcal{C}$ by a single unipotent element (contained in $\exp \mathcal{C}$) in G°, and $\mathcal{F}$ by the associated representation of the correct component group.

As new parameters we take triples (s,u,ρ) such that:

- $s \in G^\circ$ is semisimple;
- $u \in Z_G(s)^\circ$ is unipotent;
- $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(s,u))\big)$ with $q\Psi_{Z_G(s)}(u,\rho) = (M,\mathcal{C}_v^M,q\mathcal{E})$ up to G-conjugacy.

Assume that $s \in T$ and choose an algebraic homomorphism $\gamma_u : \mathrm{SL}_2(\mathbb{C}) \to Z_G(s)^\circ$ with

$$\gamma_u \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = u \quad \text{and} \quad \mathrm{d}\gamma_u \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{t}. \tag{49}$$

Using the decomposition (8) of $\mathfrak{g}$, we write

$$\vec{\gamma_u} \left(\begin{smallmatrix} \vec{z} & 0 \\ 0 & \vec{z}^{-1} \end{smallmatrix}\right) = \exp\left(\mathrm{d}\vec{\gamma_u}\begin{pmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{pmatrix}\right) \in T. \tag{50}$$

For $\vec{z} \in \mathbb{R}^d_{>0}$ we define the standard $\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})$-module

$$\bar{E}_{s,u,\rho,\vec{z}} = \mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(s|s|^{-1}),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} (\exp_{s|s|^{-1}})_* \, \mathrm{IM}^* \, E_{\log u, \mathrm{d}\vec{\gamma_u}\begin{pmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{pmatrix} - \log|s|, \log \vec{z}, \rho}.$$

and its irreducible quotient

$$\bar{M}_{s,u,\rho,\vec{z}} = \mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(s|s|^{-1}),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} (\exp_{s|s|^{-1}})_* \, \mathrm{IM}^* \, M_{\log u, \mathrm{d}\vec{\gamma_u}\begin{pmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{pmatrix} - \log|s|, \log \vec{z}, \rho}.$$

Even when $s \notin T$, the condition on ρ and [AMS2, Propositions 3.5.a and 3.7] guarantee the existence of a $g_0 \in G^\circ$ such that $g_0 s g_0^{-1} \in T$. In this case we put

$$\bar{E}_{s,u,\rho,\vec{z}} := \bar{E}_{g_0 s g_0^{-1}, g_0 u g_0^{-1}, g_0 \cdot \rho, \vec{z}} \quad \text{and} \quad \bar{M}_{s,u,\rho,\vec{z}} := \bar{M}_{g_0 s g_0^{-1}, g_0 u g_0^{-1}, g_0 \cdot \rho, \vec{z}}. \tag{51}$$

We extend the polar decomposition (31) to this setting by

$$|s| := g_0^{-1} \, |g_0 s g_0^{-1}| \, g_0.$$

With the Jordan decomposition in G° it is possible to combine s and u in a single element $g = su \in G^\circ$. Then s equals the semisimple part g_S, u becomes the unipotent part g_U and $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(g))\big)$.

Now we come to our main result about affine Hecke algebras. In the case that G is connected, it is almost the same parametrization as in [Lus4, §5.20] and [Lus5, Theorems 10.4]. The only difference is that we twist by the Iwahori–Matsumoto involution. This is necessary to improve the unsatisfactory notions of ζ-tempered and ζ-square integrable in [Lus5, Theorem 10.5].

Theorem 2.11. *Let* $\vec{z} \in \mathbb{R}^d_{>0}$.

(*a*) *The maps*

$$(g,\rho) \;\mapsto\; (s = g_S, u = g_U, \rho) \;\mapsto \bar{M}_{s,u,\rho,\vec{z}}$$

provide canonical bijections between the following sets:

- *G-conjugacy classes of pairs (g, ρ) with $g \in G^\circ$ and $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(g))\big)$ such that $q\Psi_{Z_G(g_S)}(g_U, \rho) = (M, \mathcal{C}_v^M, q\mathcal{E})$ up to G-conjugacy;*
- *G-conjugacy classes of triples (s, u, ρ) as above;*
- $\mathrm{Irr}_{\vec z}(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}))$.

(*b*) *Suppose that $s \in T$. The representations $\bar E_{s,u,\rho,\vec z}$ and $\bar M_{s,u,\rho,\vec z}$ admit the $\mathcal{O}(T)^{W_{q\mathcal{E}}}$-character $W_{q\mathcal{E}} s \vec{\gamma_u} \left(\begin{smallmatrix} \vec z & 0 \\ 0 & \vec z^{-1} \end{smallmatrix}\right)$, for a γ_u as in (49).*

(*c*) *Suppose that $\vec z \in \mathbb{R}^d_{\geq 1}$. The following are equivalent:*
- *s is contained in a compact subgroup of G°;*
- *$|s| = 1$;*
- *$\bar M_{s,u,\rho,z}$ is tempered;*
- *$\bar E_{s,u,\rho,z}$ is tempered.*

(*d*) *When $\vec z \in \mathbb{R}^d_{>1}$, $\bar M_{s,u,\rho,\vec z}$ is essentially discrete series if and only if u is distinguished in G°. In this case $|s| \in Z(G^\circ)$ and $\bar E_{s,u,\rho,\vec z} = \bar M_{s,u,\rho,\vec z}$.*

There are no essentially discrete series representations on which at least one $\mathbf{z}_j$ acts as 1.

Proof. (a) The uniqueness in the Jordan decomposition entails that the first map is a canonical bijection.

We already noted in (51) that, for every eligible triple (s, u, ρ), s lies in $\mathrm{Ad}(G^\circ)T$. Therefore we may restrict to triples with $s \in T$. Consider the map

$$(s, u, \rho) \mapsto (s, \mathcal{C}^{Z_G(s)}_{\log u}, \mathcal{F}),$$

where $\mathcal{F}$ is determined by $\mathcal{F}_{\log u} = \rho$. As in the proof of [AMS2, Corollary 3.23], this gives a canonical bijection between G-conjugacy classes of triples (s, u, ρ) and the parameters used in Lemma 2.10. Furthermore (49) just reflects (47), so Lemma 2.10 yields the desired canonical bijection with $\mathrm{Irr}_{\vec z}(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}))$.

(b) By [AMS2, (86)] the $\mathbb{H}(Z_G(s|s|^{-1}), M, q\mathcal{E}, \vec{\mathbf{r}})$-representation

$$\mathrm{IM}^* E_{\log u, \mathrm{d}\vec{\gamma_u}\left(\begin{smallmatrix} \log \vec z & 0 \\ 0 & -\log \vec z \end{smallmatrix}\right) - \log |s|, \log \vec z, \rho} \tag{52}$$

admits the central character $\big(W_{q\mathcal{E}, s|s|^{-1}}(\log |s| - \mathrm{d}\vec{\gamma_u}\left(\begin{smallmatrix} \log \vec z & 0 \\ 0 & -\log \vec z \end{smallmatrix}\right)), \log \vec z\big)$.

The element $\gamma_u \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in Z_G(s|s|^{-1})^\circ$ normalizes T, so

$$w := \gamma_u \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) M^\circ \quad \text{lies in} \quad Z_G(s|s|^{-1})^\circ \cap N_G(M^\circ)/M^\circ = (W^\circ_{\mathcal{E}})_{s|s|^{-1}} \subset W_{q\mathcal{E}, s|s|^{-1}}.$$

We can rewrite the above central character as

$$\begin{aligned} &\big(W_{q\mathcal{E}, s|s|^{-1}} w(\log |s| - \mathrm{d}\vec{\gamma_u}\left(\begin{smallmatrix} \log \vec z & 0 \\ 0 & -\log \vec z \end{smallmatrix}\right)), \log \vec z\big) = \\ &\quad \big(W_{q\mathcal{E}, s|s|^{-1}} \vec{\gamma_u} \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) (\log |s| - \mathrm{d}\vec{\gamma_u}\left(\begin{smallmatrix} \log \vec z & 0 \\ 0 & -\log \vec z \end{smallmatrix}\right)) \gamma_u \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \log \vec z\big) = \\ &\qquad \big(W_{q\mathcal{E}, s|s|^{-1}}(\log |s| + \mathrm{d}\vec{\gamma_u}\left(\begin{smallmatrix} \log \vec z & 0 \\ 0 & -\log \vec z \end{smallmatrix}\right)), \log \vec z\big). \end{aligned}$$

By Theorems 2.9.c and 2.5.c the central character of $\bar E_{s,u,\rho,\vec z}$ becomes $\big(W_{q\mathcal{E}} s \vec{\gamma_u} \left(\begin{smallmatrix} \vec z & 0 \\ 0 & \vec z^{-1} \end{smallmatrix}\right), \vec z\big)$. The same holds for the quotient $\bar M_{s,u,\rho,\vec z}$.

(c) Suppose that $s \in T$. By [AMS2, (84)] the representation (53) and its quotient

$$\mathrm{IM}^* M_{\log u, \mathrm{d}\vec{\gamma_u}\left(\begin{smallmatrix} \log \vec z & 0 \\ 0 & -\log \vec z \end{smallmatrix}\right) - \log |s|, \log \vec z, \rho} \tag{53}$$

are tempered if and only if $\log |s| \in i\mathfrak{t}_\mathbb{R}$. By definition $\log |s| \in \mathfrak{t}_\mathbb{R}$, so this condition is equivalent to $\log |s| = 0$. This is turn is equivalent to $|s| = 1$ and to $s \in T_{\rm un}$. By Theorem 2.9.d and Proposition 2.7.b this is also equivalent to temperedness of $\bar{E}_{s,u,\rho,\vec{z}}$ or $\bar{M}_{s,u,\rho,\vec{z}}$.

The proof of part (a) shows that also for general s, temperedness is equivalent to $|s| = 1$. This happens if and only if s lies in the unitary part of a torus conjugate to T, which in turn is equivalent to s lying in a compact subgroup of G°.

(d) As in part (c), it suffices to consider the case $s \in T$.

Suppose that $\bar{M}_{s,u,\rho,\vec{z}}$ is essentially discrete series. By Proposition 2.7.c and Theorem 2.9.d the representation (53) has the same property. Moreover we saw in the proof of Proposition 2.7.c that $\widetilde{Z_{G^\circ_{\rm der}}}(s|s|^{-1})^\circ$ is semisimple, so $Z_{G^\circ_{\rm der}}(s|s|^{-1})^\circ$ semisimple as well.

By assumption $\log \vec{z} \in \mathbb{R}^d_{>0}$. Now [AMS2, (85)] says that $\log u$ is distinguished in $\mathrm{Lie}\big(Z_G(s|s|^{-1})^\circ\big)$. In view of the aforementioned semisimplicity, this is the same as distinguished in $\mathfrak{g}$. So u is distinguished in G°.

Conversely, suppose that u is distinguished in G°, or equivalently that $\log u$ is distinguished in $\mathfrak{g}$. As u commutes with s, it also commutes with $|s|$ and with $s|s|^{-1}$. This implies that $R(Z_G(s|s|^{-1})^\circ, T)$ and $R(\tilde{Z}_G(s|s|^{-1})^\circ, T)$ have full rank in $R(G^\circ, T)$. By [AMS2, (85)], Theorem 2.9.d and Proposition 2.7.c $\bar{M}_{s,u,\rho,\vec{z}}$ is essentially discrete series.

Suppose that either of the above two conditions holds. Then $|s| \in T_{\rm rs}$ commutes with the distinguished unipotent element $u \in G^\circ$. This implies that the semisimple subalgebra $\mathbb{C} \log |s| \subset \mathfrak{g}$ is contained in $Z(\mathfrak{g})$. Hence $|s| \in Z(G^\circ)$. Moreover [AMS2, Theorem 3.26.b] and Lemma 1.3 imply that $\bar{E}_{s,u,\rho,\vec{z}} = \bar{M}_{s,u,\rho,\vec{z}}$.

Finally, suppose that $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ has an essentially discrete series representation on which $\mathbf{z}_j$ acts as 1. It has finite dimension, so it has an irreducible subquotient, say $\bar{M}_{s,u,\rho,\vec{z}}$. Then $\mathrm{IM}^* M_{\log u, -\log |s|, \log \vec{z}, \rho}$ is an essentially discrete series representation of $\mathbb{H}(G_{s|s|^{-1}}, M, q\mathcal{E})$, which is annihilated by $\mathbf{r}_j$. By (12) and (13) it contains a $\mathbb{H}(G_j, M_j, \mathcal{E}_j)$-representation with the same properties. But [AMS2, Theorem 3.26.c] says that this is impossible. $\square$

Let us discuss the relation between the parametrization from Theorem 2.11.a and parabolic induction. Suppose that $Q \subset G$ is an algebraic subgroup such that $Q \cap G^\circ$ is a Levi subgroup of G° and $M \subset Q$. Let (s, u, ρ) be as above, with $s, u \in Q^\circ$. Also take $\rho^Q \in \mathrm{Irr}\big(\pi_0(Z_Q(s, u))\big)$ with $q\Psi_{Z_Q(s)}(u, \rho^Q) = (M, \mathcal{C}_v^M, q\mathcal{E})$ up to Q-conjugation.

Corollary 2.12. (*a*) *There is a natural isomorphism of $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$-modules*

$$\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}) \underset{\mathcal{H}(Q,M,q\mathcal{E},\vec{\mathbf{z}})}{\otimes} \bar{E}^Q_{s,u,\rho^Q,\vec{z}} \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(Z_Q(s,u))}(\rho^Q, \rho) \otimes \bar{E}_{s,u,\rho,\vec{z}},$$

where the sum runs over all $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(s, u))\big)$ with $q\Psi_{Z_G(s)}(u, \rho) = (M, \mathcal{C}_v^M, q\mathcal{E})$ up to G-conjugation. For $\vec{z} = \vec{1}$ this isomorphism contains

$$\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}) \underset{\mathcal{H}(Q,M,q\mathcal{E},\vec{\mathbf{z}})}{\otimes} \bar{M}^Q_{s,u,\rho^Q,\vec{z}} \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(Z_Q(s,u))}(\rho^Q, \rho) \otimes \bar{M}_{s,u,\rho,\vec{z}}.$$

(*b*) *The multiplicity of $\bar{M}_{s,u,\rho,\vec{z}}$ in $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}) \underset{\mathcal{H}(Q,M,q\mathcal{E},\vec{\mathbf{z}})}{\otimes} \bar{E}^Q_{s,u,\rho^Q,\vec{z}}$ is $[\rho^Q : \rho]_{\pi_0(Z_Q(s,u))}$. It already appears that many times as a quotient, via*

$\bar{E}^Q_{s,u,\rho^Q,\vec{z}} \to \bar{M}^Q_{s,u,\rho^Q,\vec{z}}$. *More precisely, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{H}(Q,M,q\mathcal{E},\vec{\mathbf{z}})}(\bar{M}^Q_{s,u,\rho^Q,\vec{z}}, \bar{M}_{s,u,\rho,\vec{z}}) \cong \mathrm{Hom}_{\pi_0(Z_Q(s,u))}(\rho^Q,\rho)^*.$$

Proof. Recall that the analogous statement for twisted graded Hecke algebras is Proposition 1.5. To that we can apply the Iwahori–Matsumoto involution, supported by [AMS2, (83)]. Next, part (b) of Theorem 2.9 allows us to apply part (a) while retaining the desired properties. The same goes for Theorem 2.5. Then we have transferred Proposition 1.5 to the representations $\bar{E}_{s,u,\rho,\vec{z}}$ and $\bar{M}_{s,u,\rho,\vec{z}}$. $\square$

Notice that the parameters in Theorem 2.11.a do not depend on $\vec{z}$. This enables us to relate $\mathrm{Irr}_{\vec{z}}(\mathcal{H}(G,M,q\mathcal{E}))$ to an extended quotient, as in [ABPS5, §2.3] and [AMS2, (87)].

Theorem 2.13. *Let $\vec{z} \in \mathbb{R}^d_{>0}$. There exists a canonical bijection*

$$\mu_{G,M,q\mathcal{E}} : (T/\!/W_{q\mathcal{E}})_{\natural_{q\mathcal{E}}} \to \mathrm{Irr}_{\vec{z}}(\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}}))$$

such that:

- $\mu_{G,M,q\mathcal{E}}(T_{\mathrm{un}}/\!/W_{q\mathcal{E}})_{\natural_{q\mathcal{E}}} = \mathrm{Irr}_{\vec{z},\mathrm{temp}}(\mathcal{H}(G,M,q\mathcal{E}))$ *when* $\vec{z} \in \mathbb{R}^d_{\geq 1}$;
- *the central character of* $\mu_{G,M,q\mathcal{E}}(t,\pi_t)$ *is* $\big(W_{q\mathcal{E}} t \gamma \left(\begin{smallmatrix} \vec{z} & 0 \\ 0 & \vec{z}^{-1}\end{smallmatrix}\right), \vec{z}\big)$ *for some algebraic homomorphism* $\gamma : \mathrm{SL}_2(\mathbb{C}) \to Z_G(t)^\circ$.

Remark. Together with [Sol3, Theorem 5.4.2] this proves a substantial part of the ABPS conjectures [ABPS1, §15] for the twisted affine Hecke algebra $\mathcal{H}(G,M,q\mathcal{E})$. For $\vec{z} \in (0,1)^d$, $\mu_{G,M,q\mathcal{E}}(T_{\mathrm{un}}/\!/W_{q\mathcal{E}})_{\natural_{q\mathcal{E}}}$ is the anti-tempered part of $\mathrm{Irr}_{\vec{z}}(\mathcal{H}(G,M,q\mathcal{E}))$.

Proof. From Proposition 2.2 we see that

$$\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})/(\mathbf{z}_1 - 1, \ldots, \mathbf{z}_d - 1) \cong \mathcal{O}(T) \rtimes \mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}].$$

By [ABPS5, Lemma 2.3] there exists a canonical bijection

$$\begin{array}{ccc} (T/\!/W_{q\mathcal{E}})_{\natural_{q\mathcal{E}}} & \to & \mathrm{Irr}(\mathcal{O}(T) \rtimes \mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]) \\ (t,\pi_t) & \mapsto & \mathbb{C}_t \rtimes \pi_t = \mathrm{ind}_{\mathcal{O}(T)\rtimes\mathbb{C}[W_{q\mathcal{E},t},\natural_{q\mathcal{E}}]}^{\mathcal{O}(T)\rtimes\mathbb{C}[W_{q\mathcal{E}},\natural_{q\mathcal{E}}]}(\mathbb{C}_t \otimes V_{\pi_t}) \end{array}.$$

We consider $\mathbb{C}_t \rtimes \pi_t$ as an irreducible $\mathcal{H}(G,M,q\mathcal{E})$-representation with central character $(W_{q\mathcal{E}} t, 1)$. By Theorem 2.13 there exist u and ρ, unique up to $Z_G(t)$-conjugation, such that $\mathbb{C}_t \rtimes \pi_t \cong \bar{M}_{t,u,\rho,1}$. Now we define

$$\mu_{G,M,q\mathcal{E}}(t,\pi_t) = \bar{M}_{t,u,\rho,\vec{z}}.$$

This is canonical because Theorem 2.11.a is. The properties involving temperedness and the central character follow from parts (c) and (b) of Theorem 2.11. $\square$

2.3. **Comparison with the Kazhdan–Lusztig parametrization.**

Irreducible representations of affine Hecke algebras were also classified in [KaLu, Ree], in terms of equivariant K-theory. This concerns the cases with only one complex parameter $q = \mathbf{z}^2$, which is not a root of unity. In terms of Proposition 2.2 this means that $\lambda = \lambda^* = 1$. In view of (28) and [Lus2, Proposition 2.8], this happens if and only if $T = M^\circ$ is a maximal torus of G° and $v = 1$. For the upcoming comparison we assume that $M = Z_G(T)$ equals T. Then $\pi_0(Z_M(v)) = 1$, $q\mathcal{E}$ is the trivial representation and

$$\mathfrak{R}_{q\mathcal{E}} = N_G(T,B)/T \cong G/G^\circ,$$

where B is a Borel subgroup of G° containing T (called P before). The Kazhdan–Lusztig parametrization was extended to algebras of the form

$$\mathcal{H}(G,T,q\mathcal{E}=\mathrm{triv}) = \mathcal{H}(\mathcal{R}(G^\circ,T),\lambda=1,\lambda^*=1,\mathbf{z}) \rtimes \mathfrak{R}_{q\mathcal{E}}$$

in [ABPS4, §9]. The parameters are triples (t_q,u,ρ), where

- $t_q \in T$ is semisimple;
- $u \in G^\circ$ is unipotent and $t_q u t_q^{-1} = u^q$;
- $\mathcal{B}_{G^\circ}^{t_q,u}$ is the variety of Borel subgroups of G° containing t_q and u;
- $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(t_q,u))\big)$ such that every irreducible component of $\rho|_{\pi_0(Z_{G^\circ}(t_q,u))}$ appears in $H_*(\mathcal{B}_{G^\circ}^{t_q,u},\mathbb{C})$.

Two triples of this kind are considered equivalent if they are G-conjugate. The representation $\bar{M}(t_q,u,\rho)$ attached to these data is the unique irreducible quotient of the standard module

$$\bar{E}_{t_q,u,\rho} := \mathrm{Hom}_{\pi_0(Z_G(t_q,u))}\big(\rho, H_*(\mathcal{B}_{G^\circ}^{t_q,u},\mathbb{C})\big). \tag{54}$$

The classification of $\mathcal{H}(G^\circ,T,\mathcal{E}=\mathrm{triv})$ with $q=\mathbf{z}=1$ goes back to Kato [Kat, Theorem 4.1], see also [ABPS4, §8]. With [ABPS4, Remark 9.2] and the subsequent argument (which underlies the above for $q \neq 1$) it can be extended to $\mathcal{H}(G,T,q\mathcal{E}=\mathrm{triv})$. The parameters are the same as above (only with $q=1$), and the irreducible module is

$$\bar{M}(t_1,u,\rho) = \mathrm{Hom}_{\pi_0(Z_G(t_1,u))}\big(\rho, H_{d(u)}(\mathcal{B}_{G^\circ}^{t_1,u},\mathbb{C})\big), \tag{55}$$

where $d(u)$ refers to the dimension of $\mathcal{B}_{G^\circ}^{t_1,u}$ as a real variety. Clearly $\bar{M}(t_1,u,\rho)$ is again a quotient of $\bar{E}_{t_1,u,\rho}$, but for $q=1$ (54) has other irreducible quotients as well, in lower homological degree.

Lemma 2.14. *The above set of parameters (t_q,u,ρ) is naturally in bijection with the sets of parameters used in Theorem 2.11.a.*

Proof. By [ABPS4, Lemma 7.1], we obtain the same G-conjugacy classes of parameters if we replace the above t_q by a semisimple element $s \in Z_{G^\circ}(u)$. In Theorem 2.11 we also have parameters (s,u,ρ), but with a different condition on ρ, namely that

$$q\Psi_{Z_G(s)}(u,\rho) = (T, v=1, q\epsilon=\mathrm{triv}).$$

By definition this is equivalent to

$$\Psi_{Z_G(s)^\circ}(u,\rho_s) = (T, v=1, \epsilon=\mathrm{triv}), \tag{56}$$

for any irreducible constituent ρ_s of $\rho|_{\pi_0(Z_{Z_G(s)^\circ}(u))}$. Write $r = \log z \in \mathbb{R}$ and $y = \log(u) \in \mathrm{Lie}(Z_G(s))$. According to [AMS2, Proposition 3.7] for the group $Z_G(s)^\circ$, (56) is equivalent to ρ_s appearing in

$$E_{y,0,r}^\circ = \mathbb{C}_{0,r} \underset{H_*^{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y,\mathbb{C}) = H_*(\mathcal{P}_y,\mathbb{C}).$$

To make this more explicit, we assume (as we may) that $s \in T$. Then $Z_B(s) = Z_G(s)^\circ \cap B$ is a Borel subgroup of $Z_G(s)^\circ$ and

$$\begin{aligned}\mathcal{P}_y = \{gZ_B(s) \in Z_G(s)^\circ/Z_B(s) : \mathrm{Ad}(g^{-1})y \in \mathrm{Lie}(Z_B(s))\} = \\ \{gZ_B(s) \in Z_G(s)^\circ/Z_B(s) : u \in gZ_B(s)g^{-1}\} = \mathcal{B}^u_{Z_G(s)^\circ}.\end{aligned} \tag{57}$$

Hence (56) is equivalent to ρ_s appearing in $H_*(\mathcal{B}^u_{Z_G(s)^\circ},\mathbb{C})$. Let ρ° be a $\pi_0(Z_{G^\circ}(s,u))$-constituent of ρ containing ρ_s. By [ABPS4, Proposition 6.2] there are isomorphisms of $Z_{G^\circ}(s,u)$-varieties

$$\mathcal{B}^{t_q,u}_{G^\circ} \cong \mathcal{B}^{s,u}_{G^\circ} \cong \mathcal{B}^u_{Z_G(s)^\circ} \times Z_{G^\circ}(s,u)/Z_{Z_G(s)^\circ}(u). \tag{58}$$

With this and Frobenius reciprocity we see that the condition on ρ_s is also equivalent to ρ° appearing in $H_*(\mathcal{B}^{s,u}_{G^\circ},\mathbb{C})$. We conclude that the parameters (s,u,ρ) in Theorem 2.11 are equivalent to those in [ABPS4, §9], the only change being $s \leftrightarrow t_q$. $\square$

Proposition 2.15. *The parametrization of* $\mathrm{Irr}_z(\mathcal{H}(G,T,q\mathcal{E} = \mathrm{triv}))$ *obtained in Theorem 2.11.a agrees with the above parametrization by the representations* $\bar{M}(t_q,u,\rho)$*, when we set* $q = z^2 \in \mathbb{R}_{>0}$ *and take Lemma 2.14 into account. Moreover the standard modules* $\bar{E}_{s,u,\rho,z}$ *and* $\bar{E}_{t_q,u,\rho}$ *are isomorphic.*

In other words, our classification of irreducible representations of affine Hecke algebras agrees with that of Kazhdan–Lusztig and the extended versions thereof.

Remark. Our parametrization differs from the one used by Lusztig in [Lus4, §5.20] and [Lus5, Theorem 10.4], namely by the Iwahori–Matsumoto involution. Thus Proposition 2.15 shows that the classification of unipotent representations of adjoint groups in [Lus4, Lus5] does not agree with the earlier classification of Iwahori–spherical representations in [KaLu].

Proof. Let (s,u,ρ) be a triple as above, and choose an algebra homomorphism $\gamma_u : \mathrm{SL}_2(\mathbb{C}) \to Z_G(s)^\circ$ with $\gamma_u \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = u$. Then we can take $t_q = s\gamma_u \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)$, where $z^2 = q$. Recall that $\bar{M}(t_q,u,\rho)$ is a quotient of $\bar{E}_{t_q,u,\rho}$ from (54). Write $\rho = \rho^\circ \rtimes \tau^*$, where

$$\tau^* \in \mathrm{Irr}(\mathfrak{R}_{q\mathcal{E},u,s,\rho^\circ}) \quad \text{with} \quad \mathfrak{R}_{q\mathcal{E},u,s,\rho^\circ} = \pi_0(Z_G(s,u))_{\rho^\circ}/\pi_0(Z_{G^\circ}(s,u)).$$

From [ABPS4, (70)] we see that $\bar{E}_{t_q,u,\rho}$ equals

$$\mathrm{Hom}_{\pi_0(Z_{G^\circ}(s,u))}\big(\rho^\circ, H_*(\mathcal{B}^{s,u}_{G^\circ},\mathbb{C})\big) \rtimes \tau. \tag{59}$$

To the part without $\rtimes\tau$ we can apply [EvMi], which compares the two parametrizations. In [EvMi] both the Iwahori–Matsumoto involution and a related "shift" are mentioned. This involution is necessary to get temperedness for the same parameters in both classifications. Unfortunately, it is not entirely clear what Evens and Mirkovich mean by a "shift", for signs can be inserted at various places. In any case their argument is based on temperedness and a comparison of weights [EvMi, Theorem 5.5], and it will work once we arrange the modules such that these two aspects match. With this in mind, [EvMi, Theorem 6.10] says that the $\mathbb{H}(Z_{G^\circ}(s|s|^{-1}),T,\mathrm{triv})$-module obtained from $\mathrm{Hom}_{\pi_0(Z_{G^\circ}(s,u))}\big(\rho^\circ, H_*(\mathcal{B}^{s,u}_{G^\circ},\mathbb{C})\big)$ via Theorems 2.5 and 2.9 is $\mathrm{IM}^* E_{y,\mathrm{d}\gamma_u \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \log|s|, r, \rho^\circ}$. The extension with the group $\mathfrak{R}_{q\mathcal{E}}$ is handled in the same way for all algebras under consideration here, namely with Clifford theory. It follows that applying Theorems 2.5 and 2.9 to (59) yields

$$\Big(\mathrm{IM}^* E_{y,\mathrm{d}\gamma_u \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \log|s|, r, \rho^\circ}\Big) \rtimes \tau. \tag{60}$$

Moreover IM is the identity on $\mathbb{C}[\mathfrak{R}_{q\mathcal{E}}]$, so the large brackets are actually superfluous. The action of $\mathfrak{R}_{q\mathcal{E},u,s,\rho^\circ}$ underlying $\rtimes\tau$ in (59) comes from the action of $\pi_0(Z_G(s,u))$

on $H_*(\mathcal{B}^u_{Z_G(s|s|^{-1})^\circ}, \mathbb{C})$. By (57) for the group $Z_G(s|s|^{-1})^\circ$,

$$\mathcal{B}^u_{Z_G(s|s|^{-1})^\circ} = \mathcal{P}_y.$$

Via this equality the $\pi_0(Z_G(s,u))$-action on $H_*(\mathcal{B}^u_{Z_G(s|s|^{-1})^\circ}, \mathbb{C})$ agrees with the action on

$$H_*(\mathcal{P}_y, \mathbb{C}) \cong \mathbb{C}_{|s|,r} \underset{H_*^{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y, \mathbb{C})$$

from [AMS2, Theorem 3.2.d]. Hence

$$\begin{aligned}\big(\mathrm{IM}^* E_{y, \mathrm{d}\gamma_u \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \log|s|, r, \rho^\circ}\big) \rtimes \tau = \mathrm{IM}^* \big(E_{y, \mathrm{d}\gamma_u \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \log|s|, r, \rho^\circ} \rtimes \tau\big) \\ = \mathrm{IM}^* \big(E_{y, \mathrm{d}\gamma_u \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \log|s|, r, \rho^\circ \rtimes \tau *}\big) = \mathrm{IM}^* E_{y, \mathrm{d}\gamma_u \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \log|s|, r, \rho}.\end{aligned}$$

We see that the standard modules $\bar{E}_{t_q,u,\rho}$ and $\bar{E}_{s,u,\rho,z}$ give the same module upon applying Theorems 2.5 and 2.9. Hence they are isomorphic.

From here on we have to assume that $q = z^2$ is not a root of unity. We recognize the unique irreducible quotient of the right hand side as (53), a part of the definition of $\bar{M}_{s,u,\rho,z}$. Using Theorems 2.9 and 2.5 again, but now in the opposite direction, we see that both $\bar{M}_{s,u,\rho,z}$ and $\bar{M}(t_q, u, \rho)$ are the unique irreducible quotient of

$$\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(s|s|^{-1}), M, q\mathcal{E})}^{\mathcal{H}(G,M,\mathcal{E})} (\exp_{s|s|^{-1}})_* \, \mathrm{IM}^* \, E_{\log u, \mathrm{d}\gamma_u \left(\begin{smallmatrix} \log z & 0 \\ 0 & -\log z \end{smallmatrix}\right) - \log|s|, \log z, \rho}.$$

Thus the two parametrizations agree when $q = z^2 \neq 1$.

For $q = z = 1$ a different argument is needed. We note that (59) still applies, which enables us to write

$$\bar{M}(t_1 = s, u, \rho) = \mathrm{Hom}_{\pi_0(Z_{G^\circ}(s,u))} \big(\rho^\circ, H_{d(u)}(\mathcal{B}^{s,u}_{G^\circ}, \mathbb{C})\big) \rtimes \tau.$$

From the definition of the $X^*(T)$-action in [Kat, §3] we see that $H_*(\mathcal{B}^{s,u}_{G^\circ}, \mathbb{C})$ is completely reducible as a $X^*(T)$-module. With [ABPS4, Theorem 8.2] we deduce that the weight space for $s \in T$ is, as $(W_{q\mathcal{E}})_s$-representation, equal to

$$\mathrm{Hom}_{\pi_0(Z_G(s,u))} \big(\rho, H_{d(u)}(\mathcal{B}^u_{Z_G(s)^\circ}, \mathbb{C})\big) = \mathrm{Hom}_{\pi_0(Z_{Z_G(s)^\circ}(u))} \big(\rho^\circ, H_{d(u)}(\mathcal{B}^u_{Z_G(s)^\circ}, \mathbb{C})\big) \rtimes \tau.$$

From [AMS2, (39)] we can also determine the $X^*(T)$-weight space for s in $\bar{M}_{s,u,\rho,1}$. First we look at the $S(\mathfrak{t}^*)$-weight $-\log|s|$ in $M^\circ_{y,-\log|s|,0,\rho^\circ}$, that gives $M^{Q^\circ}_{y,-\log|s|,0,\rho^\circ}$. As in [AMS2, Section 3.2], we denote the underlying $W(Z_G(s)^\circ, T)$-representation by M_{y,ρ°. Next we replace $Z_G(s)^\circ$ by $Z_G(s)$ and ρ° by $\rho = \rho^\circ \rtimes \tau^*$, obtaining the $(W_{q\mathcal{E}})_s$-representation

$$M^{Q^\circ}_{y,-\log|s|,0,\rho^\circ} \rtimes \tau = M_{y,\rho^\circ} \rtimes \tau. \tag{61}$$

Applying the Iwahori–Matsumoto involution and Theorem 2.9, we get

$$(\exp_{s|s|^{-1}})_* \mathrm{IM}^* (M^{Q^\circ}_{y,-\log|s|,0,\rho^\circ} \rtimes \tau). \tag{62}$$

The previous $S(\mathfrak{t}^*)$-weight space (61) for $-\log|s|$ has now been transformed into the $X^*(T)$-weight space for s in the representation $\bar{M}_{s,u,\rho,1}$ with respect to the group $Z_G(s)$. To land inside $\bar{M}_{s,u,\rho,1}$ with respect to G, we must still apply Theorem 2.5. But that does not change the $X^*(T)$-weight space for s, so we can stick to (62).

For $r = 0, z = 1$ the map $(\exp_{s|s|^{-1}})_*$ becomes the identity on $\mathbb{C}[W_\mathcal{E}]$, see [Sol3, (2.5) and (1.25)]. It remains to compare the $\mathbb{C}[W_\mathcal{E}]$-modules

$$\mathrm{IM}^*(M_{y,\rho^\circ} \rtimes \tau) \quad \text{and} \quad \mathrm{Hom}_{\pi_0(Z_{Z_G(s)^\circ}(u))}\big(\rho^\circ, H_{d(u)}(\mathcal{B}^u_{Z_G(s)^\circ}, \mathbb{C})\big) \rtimes \tau. \tag{63}$$

By definition [AMS2, Section 3.2] M_{y,ρ° is the $W(Z_G(s)^\circ, T)$-representation associated to (y, ρ°) by the generalized Springer correspondence from [Lus1]. It differs from the classical Springer correspondence by the sign representation, so

$$M_{y,\rho^\circ} = \mathrm{sign} \; \otimes \mathrm{Hom}_{\pi_0(Z_{Z_G(s)^\circ}(u))}\big(\rho^\circ, H_{d(u)}(\mathcal{B}^u_{Z_G(s)^\circ}, \mathbb{C})\big).$$

On both sides of (63) the actions underlying $\rtimes\tau$ come from the action of $Z_G(s,u)$ on $H_*(\mathcal{B}^u_{Z_G(s)^\circ}, \mathbb{C}) \cong H_*(\mathcal{P}_u, \mathbb{C})$. Moreover $\mathrm{IM}(w) = \mathrm{sign}(w)w$ for $w \in W(Z_G(s)^\circ, T)$ and IM is the identity on the group $\mathfrak{R}$ for $Z_G(s)$. We conclude that the two representations in (63) are equal.

This proves that $\bar{M}(t_1 = s, u, \rho)$ and $\bar{M}_{s,u,\rho,1}$ have the same $X(T)$-weight space for s. Since both representations are irreducible, that implies that they are isomorphic. □

3. Langlands parameters

Let F be a local non-archimedean field and let $\mathcal{G}(F)$ be a connected reductive group over F. In this section we construct a bijection between enhanced Langlands parameters for $\mathcal{G}(F)$ and a certain collection of irreducible representations of twisted Hecke algebras.

To this end we have to collect several notions about L-parameters, for which we follow [AMS1]. For the background we refer to that paper, here we do little more than recalling the necessary notations.

Let $\mathbf{W}_F$ be the Weil group of F, $\mathbf{I}_F$ the inertia subgroup and $\mathrm{Frob}_F \in \mathbf{W}_F$ an arithmetic Frobenius element. Let $\mathcal{G}^\vee$ be the complex dual group of $\mathcal{G}$. It is endowed with an action of $\mathbf{W}_F$, which preserves a pinning of $\mathcal{G}^\vee$. The Langlands dual group is ${}^L\mathcal{G} = \mathcal{G}^\vee \rtimes \mathbf{W}_F$.

Definition 3.1. A Langlands parameter for ${}^L\mathcal{G}$ is a continuous group homomorphism

$$\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathcal{G}^\vee \rtimes \mathbf{W}_F$$

such that:

- $\phi(w) \in \mathcal{G}^\vee w$ for all $w \in \mathbf{W}_F$;
- $\phi(\mathbf{W}_F)$ consists of semisimple elements;
- $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic.

We call a L-parameter:

- bounded, if $\phi(\mathrm{Frob}_F) = (c, \mathrm{Frob}_F)$ with c in a compact subgroup of $\mathcal{G}^\vee$;
- discrete, if $Z_{\mathcal{G}^\vee}(\phi)^\circ = Z(\mathcal{G}^\vee)^{\mathbf{W}_F,\circ}$.

With [Bor, §3] it is easily seen that this definition of discreteness is equivalent to the usual involving with proper Levi subgroups.

Let $\mathcal{G}^\vee_{\mathrm{sc}}$ be the simply connected cover of the derived group $\mathcal{G}^\vee_{\mathrm{der}}$. Let $Z_{\mathcal{G}^\vee_{\mathrm{ad}}}(\phi)$ be the image of $Z_{\mathcal{G}^\vee}(\phi)$ in the adjoint group $\mathcal{G}^\vee_{\mathrm{ad}}$. We define

$$Z^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi) = \text{ inverse image of } Z_{\mathcal{G}^\vee_{\mathrm{ad}}}(\phi) \text{ under } \mathcal{G}^\vee_{\mathrm{sc}} \to \mathcal{G}^\vee.$$

Definition 3.2. To ϕ we associate the finite group $\mathcal{S}_\phi := \pi_0(Z^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi))$. An enhancement of ϕ is an irreducible representation of $\mathcal{S}_\phi$.

The group $\mathcal{G}^\vee$ acts on the collection of enhanced L-parameters for ${}^L\mathcal{G}$ by

$$g \cdot (\phi, \rho) = (g \phi g^{-1}, g \cdot \rho) \quad \text{where } g \cdot \rho(a) = \rho(g^{-1} a g) \text{ for } a \in \mathcal{S}_\phi .$$

Let $\Phi_e({}^L\mathcal{G})$ be the collection of $\mathcal{G}^\vee$-orbits of enhanced L-parameters.

Let us consider $\mathcal{G}(F)$ as an inner twist of a quasi-split group. Via the Kottwitz isomorphism it is parametrized by a character of $Z(\mathcal{G}^\vee_{\mathrm{sc}})^{\mathbf{W}_F}$, say $\zeta_\mathcal{G}$. We say that $(\phi, \rho) \in \Phi_e({}^L\mathcal{G})$ is relevant for $\mathcal{G}(F)$ if $Z(\mathcal{G}^\vee_{\mathrm{sc}})^{\mathbf{W}_F}$ acts on ρ as $\zeta_\mathcal{G}$. The subset of $\Phi_e({}^L\mathcal{G})$ which is relevant for $\mathcal{G}(F)$ is denoted $\Phi_e(\mathcal{G}(F))$.

As is well-known, $(\phi, \rho) \in \Phi_e({}^L\mathcal{G})$ is already determined by $\phi|_{\mathbf{W}_F}, u_\phi := \phi\big(1, \big(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\big)\big)$ and ρ. Sometimes we will also consider $\mathcal{G}^\vee$-conjugacy classes of such triples $(\phi|_{\mathbf{W}_F}, u_\phi, \rho)$ as enhanced L-parameters. An enhanced L-parameter $(\phi|_{\mathbf{W}_F}, v, q\epsilon)$ will often be abbreviated to $(\phi_v, q\epsilon)$. We will study enhanced Langlands parameters via their cuspidal support, as introduced in [AMS1].

Definition 3.3. For $(\phi, \rho) \in \Phi_e({}^L\mathcal{G})$ we write $G_\phi = Z^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{W}_F})$, a complex reductive group. We say that (ϕ, ρ) is cuspidal if ϕ is discrete and $\big(u_\phi = \phi\big(1, \big(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\big)\big), \rho\big)$ is a cuspidal pair for G_ϕ in the sense of [AMS1, §3]. (This means that $\rho = \mathcal{F}_{u_\phi}$, for a G_ϕ-equivariant cuspidal local system $\mathcal{F}$ on $\mathcal{C}^{G_\phi}_{u_\phi}$.) We denote the collection of cuspidal L-parameters for ${}^L\mathcal{G}$ by $\Phi_{\mathrm{cusp}}({}^L\mathcal{G})$, and the subset which is relevant for $\mathcal{G}(F)$ by $\Phi_{\mathrm{cusp}}(\mathcal{G}(F))$.

Proposition 3.4. [AMS1, Proposition 7.3]
Let $(\phi, \rho) \in \Phi_e(\mathcal{G}(F))$ and write $q\Psi_{G_\phi}(u_\phi, \rho) = [M, v, q\epsilon]_{G_\phi}$. Upon replacing (ϕ, ρ) by $\mathcal{G}^\vee$-conjugate, there exists a Levi subgroup $\mathcal{L}(F) \subset \mathcal{G}(F)$ such that $(\phi|_{\mathbf{W}_F}, v, q\epsilon)$ is a cuspidal L-parameter for $\mathcal{L}(F)$. Moreover $\mathcal{L}(F)$ is unique up to $\mathcal{G}(F)$-conjugation and

$$\mathcal{L}^\vee \rtimes \mathbf{W}_F = Z_{\mathcal{G}^\vee \rtimes \mathbf{W}_F}(Z(M)^\circ).$$

Suppose that (ϕ, ρ) is as in Proposition 3.4. We define its modified cuspidal support as

$${}^L\Psi(\phi, \rho) = (\mathcal{L}^\vee \rtimes \mathbf{W}_F, \phi|_{\mathbf{W}_F}, v, q\epsilon).$$

The right hand side consists of a Langlands dual group and a cuspidal enhanced L-parameter for that. Every enhanced L-parameter for ${}^L\mathcal{G}$ is conjugate to one as above, so the map ${}^L\Psi$ is well-defined on the whole of $\Phi_e({}^L\mathcal{G})$. Notice that ${}^L\Psi$ preserves boundedness of enhanced L-parameters.

We also need Bernstein components of enhanced L-parameters. Recall from [Hai, §3.3.1] that the group of unramified characters of $\mathcal{L}(F)$ is naturally isomorphic to $Z(\mathcal{L}^\vee \rtimes \mathbf{I}_F)^\circ_{\mathbf{W}_F}$. We consider this as an object on the Galois side of the local Langlands correspondence and we write

$$X_{\mathrm{nr}}({}^L\mathcal{L}) = Z(\mathcal{L}^\vee \rtimes \mathbf{I}_F)^\circ_{\mathbf{W}_F}.$$

Given $(\phi', \rho') \in \Phi_e(\mathcal{L}(F))$ and $z \in X_{\mathrm{nr}}({}^L\mathcal{L})$, we define $(z\phi', \rho') \in \Phi_e(\mathcal{L}(F))$ by

$$z\phi' = \phi' \text{ on } \mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C}) \text{ and } (z\phi')(\mathrm{Frob}_F) = \tilde{z}\phi'(\mathrm{Frob}_F),$$

where $\tilde{z} \in Z(\mathcal{L}^\vee \rtimes \mathbf{I}_F)^\circ$ represents z.

Definition 3.5. An inertial equivalence class for $\Phi_e(\mathcal{G}(F))$ is the $\mathcal{G}^\vee$-conjugacy class $\mathfrak{s}^\vee$ of a pair $(\mathcal{L}^\vee \rtimes \mathbf{W}_F, \mathfrak{s}_{\mathcal{L}}^\vee)$, where $\mathcal{L}(F)$ is a Levi subgroup of $\mathcal{G}(F)$ and $\mathfrak{s}_{\mathcal{L}}^\vee$ is a $X_{\mathrm{nr}}({}^L\mathcal{L})$-orbit in $\Phi_{\mathrm{cusp}}(\mathcal{L}(F))$.

The Bernstein component of $\Phi_e(\mathcal{G}(F))$ associated to $\mathfrak{s}^\vee$ is

$$\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee} := {}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, \mathfrak{s}_{\mathcal{L}}^\vee). \tag{64}$$

We denote set of inertial equivalence classes for $\Phi_e(\mathcal{G}(F))$ by $\mathfrak{B}^\vee(\mathcal{G}(F))$.

In this way, we obtain a partition of the set $\Phi_e(\mathcal{G}(F))$ analogous to the partition of $\mathrm{Irr}(\mathcal{G}(F))$ induced by its Bernstein decomposition:

$$\Phi_e(\mathcal{G}(F)) = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(\mathcal{G}(F))} \Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}, \tag{65}$$

We note that $\Phi_e(\mathcal{L}(F))^{\mathfrak{s}_{\mathcal{L}}^\vee}$ is diffeomorphic to a quotient of the complex torus $X_{\mathrm{nr}}({}^L\mathcal{L})$ by a finite subgroup, albeit not in a canonical way.

With an inertial equivalence class $\mathfrak{s}^\vee$ for $\Phi_e(\mathcal{G}(F))$ we associate the finite group

$$W_{\mathfrak{s}^\vee} := \text{ stabilizer of } \mathfrak{s}_{\mathcal{L}}^\vee \text{ in } N_{\mathcal{G}^\vee}(\mathcal{L}^\vee \rtimes \mathbf{W}_F)/\mathcal{L}^\vee. \tag{66}$$

Let $W_{\mathfrak{s}^\vee,\phi_v,q\epsilon}$ be the isotropy group of $(\phi_v, q\epsilon) \in \mathfrak{s}_{\mathcal{L}}^\vee$. With the generalized Springer correspondence [AMS1, Theorem 5.5] we can attach to any element of ${}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, \phi_v, q\epsilon)$ an irreducible projective representation of $W_{\mathfrak{s}^\vee,\phi_v,q\epsilon}$. More precisely, consider the cuspidal quasi-support

$$q\mathfrak{t} = [G_\phi \cap \mathcal{L}_c^\vee, v, q\epsilon]_{G_\phi},$$

where $\mathcal{L}_c^\vee \subset \mathcal{G}_{\mathrm{sc}}^\vee$ is the preimage of $\mathcal{L}^\vee$ under $\mathcal{H}_{\mathrm{sc}}^\vee \to \mathcal{H}^\vee$. In this setting we write the group $W_{q\mathcal{E}}$ from (5) as $W_{q\mathfrak{t}}$. By [AMS1, Lemma 8.2] $W_{q\mathfrak{t}}$ is canonically isomorphic to $W_{\mathfrak{s}^\vee,\phi_v,q\epsilon}$. According to [AMS1, Proposition 9.1] there exist a 2-cocycle $\kappa_{q\mathfrak{t}}$ of $W_{q\mathfrak{t}}$ and a bijection (canonical up to the choice of $\kappa_{q\mathfrak{t}}$ in its cohomology class)

$${}^L\Sigma_{q\mathfrak{t}} : {}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, \phi_v, q\epsilon) \to \mathrm{Irr}(\mathbb{C}[W_{q\mathfrak{t}}, \kappa_{q\mathfrak{t}}]).$$

It is given by applying the generalized Springer correspondence for $(G_\phi, q\mathfrak{t})$ to (u_ϕ, ρ).

Theorem 3.6. [AMS1, Theorem 9.3]
There exists a bijection

$$\begin{array}{ccc} \Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee} & \longleftrightarrow & \big(\Phi_e(\mathcal{L}(F))^{\mathfrak{s}_{\mathcal{L}}^\vee} /\!/ W_{\mathfrak{s}^\vee}\big)_\kappa, \\ (\phi, \rho) & \mapsto & \big({}^L\Psi(\phi, \rho), {}^L\Sigma_{q\mathfrak{t}}(\phi, \rho)\big). \end{array}$$

It is almost canonical, in the sense that it depends only the choices of 2-cocycles $\kappa_{q\mathfrak{t}}$ as above.

3.1. **Graded Hecke algebras.**

In Theorem 2.13 we saw that the irreducible representations of a (twisted) affine Hecke algebra can be parametrized with a (twisted) extended quotient of a torus by a finite group. Motivated by the analogy with Theorem 3.6, we want to associate to any Bernstein component $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$ a twisted affine Hecke algebra, whose irreducible representations are naturally parametrized by $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$. As this turns out to be complicated, we first try to do the same with twisted graded Hecke algebras. From a Bernstein component we will construct a family of algebras, such that a suitable subset of their irreducible representations is canonically in bijection with

$\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$. Of course this will be based on the cuspidal quasi-support $[M, v, q\epsilon]_{G_\phi}$ for the group

$$G_\phi := Z^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{W}_F}). \tag{67}$$

One problem is that $Z(\mathcal{G}^\vee)^\circ$ was left out of $\mathcal{G}^\vee_{\mathrm{sc}}$, so we can never see it when working in G_ϕ. We resolve this in a crude way, replacing G_ϕ by $G_\phi \times X_{\mathrm{nr}}({}^L\mathcal{G})$. Although that is not a subgroup of $\mathcal{G}^\vee$ or $\mathcal{G}^\vee_{\mathrm{sc}}$, the next result implies that the real split part of its centre is as desired.

Lemma 3.7. *We use the notations from Proposition 3.4. The natural map*

$$Z(M)^\circ \times X_{\mathrm{nr}}({}^L\mathcal{G}) \to X_{\mathrm{nr}}({}^L\mathcal{L})$$

is a finite covering of complex tori.

Proof. In Proposition 3.4 we saw that

$$\mathcal{L}^\vee \rtimes \mathbf{W}_F = Z_{\mathcal{G}^\vee \rtimes \mathbf{W}_F}(Z(M)^\circ). \tag{68}$$

Hence the image of M° under the covering $\mathcal{G}^\vee_{\mathrm{sc}} \to \mathcal{G}^\vee$ is contained in $\mathcal{L}^\vee$. It also shows that $\mathbf{W}_F$ fixes $Z(M)^\circ$ pointwise, so

$$Z(M)^\circ = (Z(M)^{\mathbf{I}_F})^\circ_{\mathbf{W}_F}.$$

As $\mathcal{L}^\vee$ is a Levi subgroup of $\mathcal{G}^\vee$, it contains $Z(\mathcal{G}^\vee)^\circ$. Hence there exists a natural map

$$Z(M)^\circ \times X_{\mathrm{nr}}(\mathcal{G}) = \big(Z(M)^{\mathbf{I}_F} \times Z(\mathcal{G}^\vee)^{\mathbf{I}_F}\big)^\circ_{\mathbf{W}_F} \to (Z(\mathcal{L}^\vee)^{\mathbf{I}_F})^\circ_{\mathbf{W}_F} = X_{\mathrm{nr}}({}^L\mathcal{L}). \tag{69}$$

The intersection of $Z(\mathcal{G}^\vee)^\circ$ and $\mathcal{G}^\vee_{\mathrm{der}}$ is finite and $Z(M)^\circ$ lands in $\mathcal{G}^\vee_{\mathrm{der}} \cap \mathcal{L}^\vee$, so the kernel of (69) is finite.

Recall from Proposition 3.4 that $\phi(\mathbf{W}_F) \subset \mathcal{L}^\vee \rtimes \mathbf{W}_F$. Hence

$$Z(\mathcal{L}^\vee \rtimes \mathbf{W}_F) \subset Z_{\mathcal{G}^\vee}(\phi(\mathbf{W}_F)) \quad \text{and} \quad Z(\mathcal{L}^\vee_c \rtimes \mathbf{W}_F)^\circ \subset Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi(\mathbf{W}_F))^\circ.$$

Since M° is a Levi subgroup of $Z_{\mathcal{G}_{\mathrm{sc}}}(\phi(\mathbf{W}_F))^\circ$ and by (68), we have $Z(M)^\circ = Z(\mathcal{L}^\vee_c \rtimes \mathbf{W}_F)^\circ$. In particular

$$\dim Z(M)^\circ = \dim Z(\mathcal{L}^\vee_c \rtimes \mathbf{W}_F)^\circ = \dim Z(\mathcal{L}^\vee_c \rtimes \mathbf{I}_F)^\circ_{\mathbf{W}_F} = \\ \dim Z(\mathcal{L}^\vee \rtimes \mathbf{I}_F)^\circ_{\mathbf{W}_F} - \dim Z(\mathcal{G}^\vee \rtimes \mathbf{I}_F)^\circ_{\mathbf{W}_F},$$

showing that both sides of (69) have the same dimension. As the map is an algebraic homomorphism between complex tori and has finite kernel, it is surjective. □

Recall that $\mathfrak{s}^\vee_{\mathcal{L}}$ came from the cuspidal quasi-support $(M, v, q\epsilon)$. For $(\phi_b|_{\mathbf{W}_F}, v, q\epsilon) \in \Phi_e(\mathcal{L}(F))^{\mathfrak{s}^\vee_{\mathcal{L}}}$ we can consider the group

$$Z^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi_b|_{\mathbf{W}_F}) \times X_{\mathrm{nr}}({}^L\mathcal{G}) = G_{\phi_b} \times X_{\mathrm{nr}}({}^L\mathcal{G}),$$

which contains $M \times X_{\mathrm{nr}}({}^L\mathcal{G})$ as a quasi-Levi subgroup. We choose an almost direct factorization for $G_\phi \times X_{\mathrm{nr}}({}^L\mathcal{G})$ as in (6) and we put

$$\begin{aligned} \mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}}) &:= \mathbb{H}\big(G_{\phi_b} \times X_{\mathrm{nr}}({}^L\mathcal{G}), M \times X_{\mathrm{nr}}({}^L\mathcal{G}), q\mathcal{E}, \vec{\mathbf{r}}\big) \\ &= \mathbb{H}\big(\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})), W_{\mathfrak{s}^\vee,(\phi_b)_v,q\epsilon}, c\vec{\mathbf{r}}, \natural_{q\mathcal{E}}\big), \end{aligned} \tag{70}$$

where $q\mathcal{E}$ is the M-equivariant cuspidal local system on $\mathcal{C}^M_{\log v}$ with $q\mathcal{E}_{\log v} = q\epsilon$ as representations of $\pi_0(Z_M(v)) = \pi_0(Z_M(\log v))$. From Lemma 3.7 we see that

$$\begin{aligned}\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}}) &= \mathbb{H}(Z^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi_b|_{\mathbf{W}_F}), M, q\mathcal{E}, \vec{\mathbf{r}}) \otimes S\big(\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{G}))^*\big) \\ &= \mathbb{H}(G_{\phi_b}, M, q\mathcal{E}, \vec{\mathbf{r}}) \otimes S\big(\mathrm{Lie}(Z(\mathcal{G}^\vee \rtimes \mathbf{I}_F)^\circ_{\mathbf{W}_F})^*\big).\end{aligned}$$

Let $X_{\mathrm{nr}}({}^L\mathcal{L}) = X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{un}} \times X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}$ be the polar decomposition of the complex torus $X_{\mathrm{nr}}({}^L\mathcal{L})$. Let $(\phi_b, v, q\epsilon) \in \Phi_e(\mathcal{L}(F))^{\mathfrak{s}^\vee_{\mathcal{L}}}$ with ϕ_b bounded. Suppose that $(\phi, \rho) \in \Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$ with:

(71)
- $\phi|_{\mathbf{I}_F} = \phi_b|_{\mathbf{I}_F}$;
- $\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1} \in X_{\mathrm{nr}}({}^L\mathcal{L}^\vee)_{\mathrm{rs}}$;
- $\mathrm{d}\phi|_{\mathrm{SL}_2(\mathbb{C})} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{Lie}(Z(M)^\circ)$.

For such (ϕ, ρ) and $\vec{r} \in \mathbb{C}^d$ we define

$$E(\phi, \rho, \vec{r}) = \mathrm{IM}^* E_{\log(u_\phi), \log(\phi(\mathrm{Frob}_F)^{-1}\phi_b(\mathrm{Frob}_F)) + \mathrm{d}\vec{\phi}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right), \vec{r}, \rho} \in \mathrm{Irr}(\mathbb{H}(\phi_b, v, q\epsilon)),$$

$$M(\phi, \rho, \vec{r}) = \mathrm{IM}^* M_{\log(u_\phi), \log(\phi(\mathrm{Frob}_F)^{-1}\phi_b(\mathrm{Frob}_F)) + \mathrm{d}\vec{\phi}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right), \vec{r}, \rho} \in \mathrm{Irr}(\mathbb{H}(\phi_b, v, q\epsilon)).$$

Theorem 3.8. *Fix $\vec{r} \in \mathbb{C}^d$ and $(\phi_b, v, q\epsilon) \in \Phi_e(\mathcal{L}(F))^{\mathfrak{s}^\vee_{\mathcal{L}}}$ with ϕ_b bounded.*

(a) *The map $(\phi, \rho) \mapsto M(\phi, \rho, \vec{r})$ defines a canonical bijection between*
- ${}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b|_{\mathbf{W}_F}, v, q\epsilon)$;
- *the irreducible representations of $\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}})$ with central character in* $\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}})/W_{\mathfrak{s}^\vee, \phi_b, v, q\epsilon} \times \{\vec{r}\}$.

(b) *Assume that $\Re(\vec{r}) \in \mathbb{R}^d_{\geq 0}$. The following are equivalent:*
- *ϕ is bounded;*
- ${}^L\Psi(\phi, \rho) = (\mathcal{L}^\vee \rtimes \mathbf{W}_F, \phi_b|_{\mathbf{W}_F}, v, q\epsilon)$;
- *$E(\phi, \rho, \vec{r})$ is tempered;*
- *$M(\phi, \rho, \vec{r})$ is tempered.*

(c) *Suppose that $\Re(\vec{r}) \in \mathbb{R}^d_{>0}$. Then ϕ is discrete if and only if $Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi(\mathbf{W}_F))^\circ$ is semisimple and $M(\phi, \rho, \vec{r})$ is essentially discrete series. In this case $M(\phi, \rho, \vec{r}) = E(\phi, \rho, \vec{r})$.*

Proof. (a) By Theorem 3.6 every element of ${}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b|_{\mathbf{W}_F}, v, q\epsilon)$ has a representative (ϕ, ρ) with $\phi|_{\mathbf{W}_F}$ in $X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b|_{\mathbf{W}_F}$. Then $\phi|_{\mathbf{I}_F}$ is fixed, so $\phi|_{\mathbf{W}_F}$ can be described by the single element $\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1} \in X_{\mathrm{nr}}({}^L\mathcal{L}^\vee)_{\mathrm{rs}}$. Since $X_{\mathrm{nr}}({}^L\mathcal{L}^\vee)_{\mathrm{rs}}$ is the real split part of a complex torus, there is a unique logarithm

$$\sigma_0 = \log\big(\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1}\big) \in \mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L}^\vee)_{\mathrm{rs}}). \tag{72}$$

Clearly (ϕ_b, v) is the unique bounded L-parameter in $X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}(\phi_b, v)$. This implies

$$G_\phi = Z^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{W}_F}) \subset Z^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi_b|_{\mathbf{W}_F}) = G_{\phi_b}.$$

In particular $\phi(\mathrm{SL}_2(\mathbb{C})) \subset G_{\phi_b}$ and

$$\pi_0(Z_{G_\phi}(u_\phi)) = \pi_0(Z_{G_{\phi_b}}(\sigma, u_\phi)).$$

By assumption $q\Psi_{G_\phi}(u_\phi, \rho) = (v, q\epsilon)$, and by [AMS2, Proposition 3.7] this cuspidal quasi-support is relevant for

$$\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}}) = \mathbb{H}(G_{\phi_b} \times X_{\mathrm{nr}}({}^L\mathcal{G}), M \times X_{\mathrm{nr}}({}^L\mathcal{G}), q\mathcal{E}, \vec{\mathbf{r}}).$$

According to [AMS2, Proposition 3.5.c], (ϕ, ρ) is conjugate to an enhanced L-parameter with all the above properties, which in addition satisfies

$$\mathrm{d}\phi|_{\mathrm{SL}_2(\mathbb{C})} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{Lie}(Z(M)^\circ).$$

Consequently $(\log(u_\phi), \sigma_0, \vec{r}, \rho)$ is a parameter of the kind considered in Section 1, and $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ can play the role of $\vec{\gamma}$ from (14).

Conversely, by reversing the above procedure every parameter $(y, \sigma', \vec{r}, \rho')$ for $\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}})$ gives rise to an element of ${}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}} \phi_b|_{\mathbf{W}_F}, v, q\epsilon)$. The equivalence relations on these two sets of parameters agree, for both come from conjugation by G_{ϕ_b}.

Now Theorem 1.4 says that ${}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}} \phi_b|_{\mathbf{W}_F}, v, q\epsilon)$ parametrizes the part of $\mathrm{Irr}_r(\mathbb{H}(\phi_b, v, q\epsilon))$ with central character in $\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}})/W_{\mathfrak{s}^\vee,\phi_b,v,q\epsilon} \times \{\vec{r}\}$. As in [AMS2, Theorem 3.29] and Proposition 1.6, we compose this parametrization with the Iwahori–Matsumoto involution from (25). Then the representation associated to (ϕ, ρ) becomes $\pi(\phi, \rho, r)$.

(b) By [AMS2, Theorem 3.25] and [AMS2, (84)] the third and the fourth statements are both equivalent to

$$\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1} \in \mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{un}}).$$

But by construction this is an element of $\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}})$, so the statement becomes $\phi(\mathrm{Frob}_F) = \phi_b(\mathrm{Frob}_F)$. As (ϕ_b, v) is the only bounded L-parameter in $X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}(\phi_b, v)$, this holds if and only if ϕ is bounded. Since the map ${}^L\Psi$ preserves $\phi|_{\mathbf{W}_F}$, the statement $\phi(\mathrm{Frob}_F) = \phi_b(\mathrm{Frob}_F)$ is also equivalent to

$${}^L\Psi(\phi, \rho) = (\mathcal{L}^\vee \rtimes \mathbf{W}_F, \phi_b|_{\mathbf{W}_F}, v, q\epsilon).$$

(c) Suppose that ϕ is discrete. Then

$$G_\phi^\circ = Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi(\mathbf{W}_F))^\circ = Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi_b(\mathbf{W}_F), \sigma)^\circ$$

is a reductive group in which $\phi(\mathrm{SL}_2(\mathbb{C}))$ has finite centralizer. This implies that G_ϕ° is semisimple and that u_ϕ is distinguished in it. The first of these two properties implies that G_ϕ° is a full rank subgroup of G_{ϕ_b}, and that $G_{\phi_b}^\circ$ is also semisimple. Then u_ϕ is distinguished in $G_{\phi_b}^\circ$ as well, and [AMS2, (85)] says that $M(\phi, \rho, \vec{r})$ is essentially discrete series.

Conversely, suppose that $M(\phi, \rho, \vec{r})$ is essentially discrete series and that G_ϕ° is semisimple. In the same way we deduce that $G_{\phi_b}^\circ$ is also semisimple, and that G_ϕ° has full rank in $G_{\phi_b}^\circ$. By [AMS2, (85)] $u_\phi \in G_\phi^\circ$ is distinguished in $G_{\phi_b}^\circ$, so it is also distinguished in G_ϕ°. Hence $Z_{G_\phi}(u_\phi)^\circ$ is unipotent. It is known (see for example [Ree, §4.3]) that $Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi)^\circ = Z_{G_\phi}\big(\phi(\mathrm{SL}_2(\mathbb{C}))\big)^\circ$ is the maximal reductive quotient of $Z_{G_\phi}(u_\phi)^\circ$. Hence $Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi)^\circ$ is trivial, which means that ϕ is discrete.

The final claim follows from Proposition 1.6.c. $\square$

We conclude this paragraph with some remarks about parabolic induction. Suppose that $\mathcal{Q}(F) \subset \mathcal{G}(F)$ is a Levi subgroup such that ϕ has image in ${}^L\mathcal{Q}$. Let $\mathcal{Q}_c^\vee$ be the inverse image of $\mathcal{H}^\vee$ in $\mathcal{G}^\vee_{\mathrm{sc}}$, by [Bor, §3] it equals $Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(Z(\mathcal{Q}_c^\vee \rtimes \mathbf{W}_F)^\circ)$. Therefore

(73)
$$Z^1_{\mathcal{Q}_c^\vee}(\phi_b|_{\mathbf{W}_F}) = Z^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi_b|_{\mathbf{W}_F}) \cap Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(Z(\mathcal{Q}_c^\vee \rtimes \mathbf{W}_F)^\circ) = G_{\phi_b} \cap Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(Z(\mathcal{Q}_c^\vee \rtimes \mathbf{W}_F)^\circ),$$

which shows that

$$G_{\phi_b}^\circ \cap Z^1_{\mathcal{Q}_c^\vee}(\phi_b|_{\mathbf{W}_F}) = Z_{\mathcal{Q}_c^\vee}(\phi_b(\mathbf{W}_F))^\circ$$

is a Levi subgroup of $G_{\phi_b}^\circ$. Moreover $Z^1_{\mathcal{Q}_c^\vee}(\phi_b|_{\mathbf{W}_F})$ contains M, for the cuspidal quasi-support of (ϕ,ρ) with respect to ${}^L\mathcal{G}$ is the same as the cuspidal quasi-support of (ϕ,ρ^Q) with respect to ${}^L\mathcal{Q}$, for a suitable ρ^Q [AMS1, Propostion 5.6.a].

That is, $G_{\phi_b} \times X_{\mathrm{nr}}({}^L\mathcal{G})$ and $Z^1_{\mathcal{Q}_c^\vee}(\phi_b|_{\mathbf{W}_F}) \times X_{\mathrm{nr}}({}^L\mathcal{G})$ fulfill the conditions of [AMS2, Proposition 3.22] and Corollary 2.12. It follows that the families of representations

$$(\phi,\rho,\vec{r}) \mapsto E_{\log(u_\phi),\log(\phi(\mathrm{Frob}_F)^{-1}\phi_b(\mathrm{Frob}_F))+\mathrm{d}\vec{\phi}\left(\begin{smallmatrix}\vec{r} & 0\\ 0 & -\vec{r}\end{smallmatrix}\right),\vec{r},\rho} \in \mathrm{Mod}(\mathbb{H}(\phi_b,v,q\epsilon,\vec{\mathbf{r}})),$$
$$(\phi,\rho,\vec{r}) \mapsto M_{\log(u_\phi),\log(\phi(\mathrm{Frob}_F)^{-1}\phi_b(\mathrm{Frob}_F))+\mathrm{d}\vec{\phi}\left(\begin{smallmatrix}\vec{r} & 0\\ 0 & -\vec{r}\end{smallmatrix}\right),\vec{r},\rho} \in \mathrm{Irr}(\mathbb{H}(\phi_b,v,q\epsilon,\vec{\mathbf{r}}))$$

are compatible with parabolic induction in the same sense as [AMS2, Proposition 3.22] and Corollary 2.12. In view of [AMS2, (83)] this does not change upon applying the Iwahori–Matsumoto involution, so it also goes for the representations $E(\phi,\rho,\vec{r})$ and $M(\phi,\rho,\vec{r})$ considered in Theorem 3.8.

3.2. **Affine Hecke algebras.**

Let us fix an inertial equivalence class $\mathfrak{s}^\vee$ for $\Phi_e(G^\vee)$, and use the notations from Proposition 3.4. For any $(\phi|_{\mathbf{W}_F},v,q\epsilon) \in \mathfrak{s}^\vee_{\mathcal{L}}$ we define

$$J := Z^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{I}_F}), \tag{74}$$

with G_ϕ as in (67). The groups J and G_ϕ are possibly disconnected reductive groups with $J \supset G_\phi$.

Proposition 3.9. *Define $R(J^\circ,T)$ as the set of $\alpha \in X^*(T) \setminus \{0\}$ which appear in the adjoint action of T on Lie (J°).*

(a) $R(J^\circ,T)$ is a root system.

(b) There exists a $(\phi_1|_{\mathbf{W}_F},v,q\epsilon)$ such that $R(G^\circ_{\phi_1},T) = R(J^\circ,T)$.

Remark. This result does not imply that $G^\circ_{\phi_1} = J^\circ$, as one can easily see in examples.

Proof. (a) From [AMS1, Lemma 1.1.a] we know that every $R(G^\circ_\phi,T)$ is a root system. However, this result does not apply to our current J°, as $(M,v,q\epsilon)$ need not be cuspidal quasi-support for a group with neutral component J°.

We will check the axioms of a root system for $R(J^\circ,T)$. For arbitrary $\alpha,\beta \in R(J^\circ,T)$, we have to show that

(1) $\langle \alpha^\vee , \beta \rangle \in \mathbb{Z}$;
(2) $s_\alpha(\beta) \in R(J^\circ,T)$, where $s_\alpha : X^*(T) \to X^*(T)$ is the reflection associated to α and $\alpha^\vee$.

Assume first that α and β are linearly independent in $X^*(T)$. The element $\phi(\mathrm{Frob}_F) \in \mathcal{L}^\vee \rtimes \mathbf{W}_F$ centralizes T and normalizes J°, so it stabilizes each of the root subspaces $\mathfrak{g}_\alpha \subset \mathrm{Lie}(J^\circ)$. Let λ_α (respectively λ_β) be an eigenvalue of $\mathrm{Ad}(\phi(\mathrm{Frob}_F))|_{\mathfrak{g}_\alpha}$ (respectively $\mathrm{Ad}(\phi(\mathrm{Frob}_F))|_{\mathfrak{g}_\beta}$). Since α and β are linearly independent, we can find a $t \in T$ with $\alpha(t^{-1}) = \lambda_\alpha$ and $\beta(t^{-1}) = \lambda_\beta$. Define $(\phi_t|_{\mathbf{W}_F},v,q\epsilon) \in \mathfrak{s}^\vee_{\mathcal{L}}$ by $\phi_t|_{\mathbf{I}_F} = \phi|_{\mathbf{I}_F}$ and

$$\phi_t(\mathrm{Frob}_F) = \phi(\mathrm{Frob}_F)(\text{image of } t \text{ in } \mathcal{G}^\vee_{\mathrm{der}}). \tag{75}$$

Clearly $\alpha,\beta \in R(G^\circ_{\phi_t},T)$. Since this is a root system, (i) and (ii) hold for α and β inside $R(G^\circ_{\phi_t},T)$. Then they are also valid in the larger set $R(J^\circ,T)$.

Next we consider linearly dependent α, β. Then $s_\alpha(\beta) = -\beta$, so (ii) is automatically fulfilled.

Suppose that there exists a $\gamma \in R(J^\circ, T) \setminus \mathbb{Q}\alpha$ which is not orthogonal to α. As before, we can find ϕ_2, ϕ_3 such that $\alpha, \gamma \in R(G^\circ_{\phi_2}, T)$ and $\beta, \gamma \in R(G^\circ_{\phi_3}, T)$. Hence both $\{\alpha, \gamma\}$ and $\{\beta, \gamma\}$ generate rank two irreducible root systems in $X^*(T)$, and these root systems have the same $\mathbb{Q}$-span. From the classification of rank two root systems we see that $\mathbb{Q}\alpha \cap R(J^\circ, T)$ is either $\{\pm\tilde\alpha\}$ or $\{\pm\tilde\alpha, \pm 2\tilde\alpha\}$ for a suitable $\tilde\alpha$. In particular (i) holds, for

$$\langle \alpha^\vee , \beta \rangle \in \pm\{1, 2, 4\} \subset \mathbb{Z}.$$

Finally we suppose that $\mathbb{Q}\alpha \cap R(J^\circ, T)$ is orthogonal to $R(J^\circ, T) \setminus \mathbb{Q}\alpha$. The Lie algebra of $Z(M)^\circ = T$ admits a decomposition $\mathfrak{t} = \ker \alpha \oplus \mathfrak{t}_\alpha$, such that all roots in $R(J^\circ, T) \setminus \mathbb{Q}\alpha$ vanish on $\mathfrak{t}_\alpha$. Write $J_\alpha = Z_J(\ker \alpha)$, so J°_α is a Levi subgroup of J° and $(M, \mathcal{C}^M_v, q\mathcal{E})$ is a cuspidal quasi-support for $J_\alpha \cap G_\phi$. Also $(M^\circ, \mathcal{C}^{M^\circ}_v, \mathcal{L})$ is a cuspidal support for $J^\circ_\alpha \cap G^\circ_\phi$.

Choose ϕ such that $\mathbb{Q}\alpha \cap R(G^\circ_\phi, T)$ is nonempty. Then $R(J^\circ_\alpha \cap G^\circ_\phi, T)$ is a rank one root system. This imposes strong restrictions on M° and T. All the remaining possibilities are analysed in the proof of [Lus2, Proposition 2.8]. From Lusztig's list one sees that $\mathrm{Lie}(M^\circ)$ determines $\mathrm{Lie}(J^\circ_\alpha \cap G^\circ_\phi)$. In particular $R(J^\circ_\alpha \cap G^\circ_\phi, T)$ is the same for all ϕ with $\mathbb{Q}\alpha \cap R(G^\circ_\phi, T) \neq \emptyset$. Therefore $\mathbb{Q}\alpha \cap R(J^\circ, T)$ equals the rank one root system $R(J^\circ_\alpha \cap G^\circ_\phi, T)$, and it satisfies axiom (i).

(b) Let Δ be a basis of the reduced root system $R(J^\circ, T)_{\mathrm{red}}$ – which is well-defined by part (a). Let $\lambda_\alpha \in \mathbb{C}$ $(\alpha \in \Delta)$ be an eigenvalue of $\mathrm{Ad}(\phi(\mathrm{Frob}_F))$ on $\mathfrak{g}_\alpha$. Since Δ is linearly independent, we can find $t_1 \in T$ with $\alpha(t_1^{-1}) = \lambda_\alpha$ for all $\alpha \in \Delta$. We put $\phi_1 := \phi_{t_1}$, where ϕ_{t_1} is defined by (75). Then Δ is contained in the root system $R(G^\circ_{\phi_1}, T)$. The Weyl group of (J°, T) is generated by the reflections s_α with $\alpha \in \Delta$, so it equals the Weyl group of $(G^\circ_{\phi_1}, T)$. In particular it stabilizes $R(G^\circ_{\phi_1}, T)$. Every element of $R(J^\circ, T)_{\mathrm{red}}$ is in the Weyl group orbit of some $\alpha \in \Delta$, so $R(G^\circ_{\phi_1}, T)$ contains $R(J^\circ, T)_{\mathrm{red}}$.

Suppose now that $\alpha, 2\alpha \in R(J^\circ, T)$. As above and as in [Lus2, Proposition 2.8], we consider $J^\circ_\alpha = Z_{J^\circ}(\ker \alpha)$. we saw that $\mathrm{Lie}(M^\circ)$ determines $\mathrm{Lie}(J^\circ_\alpha \cap G^\circ_\phi)$ whenever $R(G^\circ_\phi, T) \cap \mathbb{Q}\alpha$ is nonempty. Consequently $2\alpha \in R(J^\circ_\alpha \cap G^\circ_\phi, T)$ for any such ϕ. In particular $2\alpha \in R(G^\circ_{\phi_1}, T)$ whenever $\alpha, 2\alpha \in R(J^\circ, T)$. As $R(G^\circ_{\phi_1}, T) \supset R(J^\circ, T)_{\mathrm{red}}$, this means that $R(G^\circ_{\phi_1}, T) = R(J^\circ, T)_{\mathrm{red}}$. □

We define

$$W^\circ_{\mathfrak{s}^\vee} := W(R(J^\circ, T)) = N_{J^\circ}(T)/Z_{J^\circ}(T). \tag{76}$$

Since $\mathcal{L}^\vee_c = Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(T)$, it equals

$$N_{Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi(\mathbf{I}_F))}(T)/Z_{\mathcal{L}^\vee_c}(\phi(\mathbf{I}_F))^\circ = N_{J^\circ}(\mathcal{L}^\vee_c)/Z_{J^\circ}(\mathcal{L}^\vee_c).$$

By Proposition 3.9, (76) also equals

$$W(R(G^\circ_{\phi_1}, T)) = N_{G^\circ_{\phi_1}}(T)/Z_{G^\circ_{\phi_1}}(T).$$

Any element of $G^\circ_{\phi_1}$ which normalizes $T = (Z(M)^\circ)^{\mathbf{W}_F}$ will also normalize $\mathcal{L}^\vee \rtimes \mathbf{W}_F = Z_{\mathcal{G}^\vee \rtimes \mathbf{W}_F}(T)$ and $M = Z_{G_{\phi_1}}(T)$, while by [AMS2, Lemma 2.1] it stabilizes $\mathcal{C}^M_v$ and $q\mathcal{E}$. The group

$$W_{\mathfrak{s}^\vee} \subset N_{\mathcal{G}^\vee}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, M, q\mathcal{E})/\mathcal{L}^\vee$$

from (66) stabilizes the $\mathcal{L}^\vee$-conjugacy class of $X_{\rm nr}(\mathcal{L}^\vee)(\phi|_{\mathbf{W}_F}, v, q\epsilon)$, so we can represent it with elements that preserve $\phi|_{\mathbf{I}_F}$ and normalize $J = Z^1_{\mathcal{H}^\vee_{\rm sc}}(\phi|_{\mathbf{I}_F})$, M and T. As $\mathcal{L}^\vee$ centralizes T, $W_{\mathfrak{s}^\vee}$ naturally contains $W^\circ_{\mathfrak{s}^\vee}$, and it acts on $R(J^\circ, T)$.

Let $R^+(J^\circ, T)$ be the positive system defined by a parabolic subgroup $P^\circ \subset J^\circ$ with Levi factor M°. By Proposition 3.9.a any two such P° are J°-conjugate, so the choice is inessential. Since $W_{\mathfrak{s}^\vee}$ acts simply transitively on the collection of positive systems for $R(J^\circ, T)$, we obtain a semi-direct factorization

$$\begin{aligned} W_{\mathfrak{s}^\vee} &= W^\circ_{\mathfrak{s}^\vee} \rtimes \mathfrak{R}_{\mathfrak{s}^\vee}, \\ \mathfrak{R}_{\mathfrak{s}^\vee} &= \{w \in W_{\mathfrak{s}^\vee} : w R^+(J^\circ, T) = R^+(J^\circ, T)\}. \end{aligned}$$

We choose a ϕ_1 as in Proposition 3.9, which will play the role of a basepoint on $\mathfrak{s}^\vee_{\mathcal{L}}$. Then

$$W(R(G^\circ_{\phi_1}, T)) \cong (W_{\mathfrak{s}^\vee, \phi_1, v, q\epsilon})^\circ = W^\circ_{\mathfrak{s}^\vee},$$

but the group $\mathfrak{R}_{\mathfrak{s}^\vee}$ need not fix $(\phi_1|_{\mathbf{W}_F}, v, q\epsilon)$. Clearly the set

$$X_{\rm nr}({}^L\mathcal{L})_{\mathfrak{s}^\vee} := \{z \in X_{\rm nr}({}^L\mathcal{L}) : z\phi_1 \equiv_{\mathcal{L}^\vee} \phi_1\}$$

only depends on $\mathfrak{s}^\vee_{\mathcal{L}}$, not on ϕ_1. Moreover it is finite, for it consists of elements coming from the finite group $\mathcal{L}^\vee_{\rm der} \cap Z(\mathcal{L}^\vee)$. Writing

$$T_{\mathfrak{s}^\vee} = X_{\rm nr}({}^L\mathcal{L}) / X_{\rm nr}({}^L\mathcal{L})_{\mathfrak{s}^\vee},$$

we obtain a bijection

$$T_{\mathfrak{s}^\vee} \to \mathfrak{s}^\vee_{\mathcal{L}} : z \mapsto [z\phi_1|_{\mathbf{W}_F}, v, q\epsilon]. \tag{77}$$

Via this bijection we can retract the action of $W_{\mathfrak{s}^\vee}$ on $X_{\rm nr}({}^L\mathcal{L})$ to $T_{\mathfrak{s}^\vee}$. Then $W^\circ_{\mathfrak{s}^\vee}$ fixes $1 \in T_{\mathfrak{s}^\vee}$. If ϕ_0 is another basepoint, like ϕ_1, then also $W(R(G^\circ_{\phi_0}, T)) \cong W^\circ_{\mathfrak{s}^\vee}$, so $t_0 \in T^{W^\circ_{\mathfrak{s}^\vee}}$. Consequently the action of $W^\circ_{\mathfrak{s}^\vee}$ on T is independent of the choice of ϕ_1. In the other hand, the action of $\mathfrak{R}_{\mathfrak{s}^\vee}$ on $T_{\mathfrak{s}^\vee}$ may very well depend on the choice of the basepoint ϕ_1.

The elements of $R(J^\circ, T)$ extend to characters of $T \times X_{\rm nr}({}^L\mathcal{H})$, trivial on the second factor.

Lemma 3.10. *The subset $R(J^\circ, T) \subset X^*(T \times X_{\rm nr}({}^L\mathcal{H}))$ naturally descends to a root system in $X^*(T_{\mathfrak{s}^\vee})$.*

Proof. As these roots come from the adjoint action of T on Lie(J°), they are trivial on central elements. Then Lemma 3.7 shows that $R(J^\circ, T)$ naturally descends to a set of algebraic characters of $X_{\rm nr}({}^L\mathcal{L})$, namely those appearing in the adjoint action of $Z(\mathcal{L} \rtimes \mathbf{I}_F)^\circ$ on Lie(J°).

Consider any $z \in X_{\rm nr}({}^L\mathcal{L})_{\mathfrak{s}^\vee}$. Then $G_{z\phi_1} = G_{\phi_1}$ so $\alpha(z) = 1$ for any $\alpha \in R(J^\circ, T)$. Now $R(J^\circ, T)$ descends to a root system in $X^*(T_{\mathfrak{s}^\vee})$ via (77). □

We endow the root datum

$$\mathcal{R}_{\mathfrak{s}^\vee} := \big(R(J^\circ, T), X^*(T_{\mathfrak{s}^\vee}), R(J^\circ, T)^\vee, X_*(T_{\mathfrak{s}^\vee}) \big)$$

with the set of simple roots determined by a parabolic subgroup $P^\circ \subset J^\circ$. We want to define parameter functions λ and λ^* for $\mathcal{R}_{\mathfrak{s}^\vee}$, so that for every $(\phi_b|_{\mathbf{W}_F}, v, q\epsilon) \in \mathfrak{s}^\vee_{\mathcal{L}}$ with ϕ_b bounded, the reduction to graded Hecke algebras via Theorem 2.9 gives the algebra $\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}})$ from (70). For $\phi_b = \phi_1$ (43) imposes the conditions

$$\begin{array}{ll} 2\lambda(\alpha) = 2\lambda^*(\alpha) = c(\alpha) & \alpha \in R(J^\circ, T), \alpha^\vee \notin 2X_*(T_{\mathfrak{s}^\vee}), \\ \lambda(\alpha) + \lambda^*(\alpha) = c(\alpha) & \alpha \in R(J^\circ, T), \alpha^\vee \in 2X_*(T_{\mathfrak{s}^\vee}), \end{array} \tag{78}$$

where $c(\alpha)$ is computed as in Proposition 2.1, with respect to $G^\circ_{\phi_1}$. When $\alpha^\vee \notin 2X_*(T_{\mathfrak s^\vee})$, every $\phi_b = t\phi_1$ with t fixed by s_α satisfies $t(\alpha) = 1$, so for $G^\circ_{\phi_b}$ we get the same value of $c(\alpha)$.

This is a little more complicated if $\alpha^\vee \in 2X_*(T_{\mathfrak s^\vee})$. Then any $\phi_b = t\phi_1$ with t fixed by s_α has $\alpha(t) \in \{1, -1\}$. Whenever $\alpha(t) = 1$, ϕ_b gives the same $c(\alpha)$ as ϕ_1. When $\alpha(t) = -1$, (43) imposes the new condition

$$\lambda(\alpha) - \lambda^*(\alpha) = c^*(\alpha) := \left\{ \begin{array}{ll} c_t(\alpha) & \alpha \in R(G^\circ_{t\phi_1}, T) \\ c_t(2\alpha)/2 & \alpha \notin R(G^\circ_{t\phi_1}, T) \end{array} \right. \tag{79}$$

Here c_t means c computed with respect to $G^\circ_{t\phi_1}$. Clearly the system of equations (78) and (79) has unique solutions $\lambda(\alpha), \lambda^*(\alpha) \in \mathbb{Q}$.

Lemma 3.11. *There exists a basepoint ϕ_1 such that $\lambda(\alpha), \lambda^*(\alpha) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in R(J^\circ, T)$.*

Proof. There only is an issue when $\alpha^\vee \in 2X_*(T_{\mathfrak s^\vee})$. Notice that $c^*(\alpha) \in \mathbb{Z}$ by Proposition 2.1. With [Lus2] we can compare the parities of $c(\alpha)$ and $c^*(\alpha)$.

(i) If $\alpha \in R(G^\circ_{t\phi_1}, T), 2\alpha \in R(G^\circ_{\phi_1}, T)$, then $2\alpha \in R(G^\circ_{t\phi_1}, T)$. By [Lus2, Proposition 2.8] $c(\alpha)$ and $c^*(\alpha)$ are both even.
(ii) If $\alpha \in R(G^\circ_{t\phi_1}, T)$ and $2\alpha \notin R(G^\circ_{\phi_1}, T) = R(J^\circ, T)$, then $2\alpha \notin R(G^\circ_{t\phi_1}, T) \subset R(J^\circ, T)$. By [Lus2, Proposition 2.8] $c(\alpha)$ and $c^*(\alpha)$ are both odd.
(iii) If $2\alpha \in R(G^\circ_{t\phi_1}, T)$ but $\alpha \notin R(G^\circ_{t\phi_1}, T)$ then $R(J^\circ, T)$ has type BC_k. For a suitable choice of t, $R(G^\circ_{t\phi_1}, T)$ has type C_k. By [Lus2, §2.13] $c(\alpha)$ is odd and $c^*(\alpha) = c_t(2\alpha)/2 = 1$.

In the cases (i) and (ii) it may occur that $c^*(\alpha) > c(\alpha)$. When that happens, we consider the unique simple root α' in the same $W_{\mathfrak s^\vee}$-orbit as α, and we choose $t' \in T$ with $\alpha'(t') = -1$ and $\beta(t') = 1$ for all other simple roots. Then we replace ϕ_1 by $t'\phi_1$, which means that we exchange $c(\alpha') = c(\alpha)$ with $c^*(\alpha) = c^*(\alpha')$ and leave all other parameters as they were.

With this and the above parity comparison we get

$$\lambda(\alpha) = (c(\alpha) + c^*(\alpha))/2 \in \mathbb{Z}_{>0} \quad \text{and} \quad \lambda^*(\alpha) = (c(\alpha) - c^*(\alpha))/2 \in \mathbb{Z}_{\geq 0}. \qquad \square$$

To $\mathfrak s^\vee$ we can associate the affine Hecke algebra $\mathcal{H}(\mathcal{R}_{\mathfrak s^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$, where ϕ_1 is as in Lemma 3.11 and λ and λ^* satisfy (78) and (79). However, this algebra takes only the subgroup $W^\circ_{\mathfrak s^\vee}$ of $W_{\mathfrak s^\vee}$ into account. To see $W_{\mathfrak s^\vee, \phi_1, v, q\epsilon}$, we can enlarge it to

$$\begin{aligned} & \mathcal{H}(\mathcal{R}_{\mathfrak s^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak s^\vee, \phi_1, v, q\epsilon}, \natural_{\mathfrak s^\vee, \phi_1, v, q\epsilon}] = \\ & \mathcal{H}(\mathcal{R}_{\mathfrak s^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathrm{End}^+_{\mathcal{D}\mathrm{Lie}(G_{\phi_1})_{\mathrm{RS}}}\big((\mathrm{pr}_1)_! \dot{q\mathcal{E}}\big). \end{aligned} \tag{80}$$

But $W_{\mathfrak s^\vee}$ can also contain elements that do not fix ϕ_1. In fact, in some cases $W_{\mathfrak s^\vee}$ even acts freely on $T_{\mathfrak s^\vee}$.

Proposition 3.12. *Assume that the almost direct factorization (6) of J° is $W_{\mathfrak s^\vee}$-stable.*

(*a*) *The group $\mathfrak{R}_{\mathfrak s^\vee}$ acts naturally on $\mathcal{H}(\mathcal{R}_{\mathfrak s^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$, by algebra automorphisms.*
(*b*) *This can be realized in a twisted affine Hecke algebra*

$$\mathcal{H}(\mathcal{R}_{\mathfrak s^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak s^\vee}, \natural_{\mathfrak s^\vee}] = \mathcal{H}(\mathcal{R}_{\mathfrak s^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathrm{End}^+_{\mathcal{D}\mathrm{Lie}(J)_{\mathrm{RS}}}\big((\mathrm{pr}_1)_! \dot{q\mathcal{E}}\big)$$

in which (80) is canonically embedded.

Proof. (a) The action of $\mathfrak{R}_{\mathfrak{s}^\vee}$ on $T_{\mathfrak{s}^\vee}$ comes from (77). This determines an action on $\mathcal{O}(T_{\mathfrak{s}^\vee}) \cong \mathbb{C}[X^*(T_{\mathfrak{s}^\vee})]$. Any $\gamma \in \mathfrak{R}_{\mathfrak{s}^\vee}$ maps θ_x to an invertible element of $\mathbb{C}[X^*(T_{\mathfrak{s}^\vee})]$. That is,

$$\gamma \cdot \theta_x = \theta_{\gamma x} \lambda_{\gamma,x} \text{ with } \lambda_{\gamma,x} \in \mathbb{C}^\times .$$

The linear part $x \mapsto \gamma x$ is an automorphism of $X^*(T_{\mathfrak{s}^\vee})$, and the translation part of $\gamma : T_{\mathfrak{s}^\vee} \to T_{\mathfrak{s}^\vee}$ is given by $\lambda_{\gamma,x}^{-1} = x(\gamma(1))$. Since $W_{\mathfrak{s}^\vee}^\circ$ is normal in $W_{\mathfrak{s}^\vee}$,

$$(W_{\mathfrak{s}^\vee}^\circ)_{\gamma(1)} = \big(\gamma W_{\mathfrak{s}^\vee}^\circ \gamma^{-1}\big)_1 = (W_{\mathfrak{s}^\vee}^\circ)_1 = W_{\mathfrak{s}^\vee}^\circ .$$

In other words, the translation part of γ commutes with all the reflections s_α ($\alpha \in R(J^\circ, T)$).

According to [AMS1, lemma 9.2] there exists a canonical algebra isomorphism

$$\psi_{\gamma,\phi_1,v,q\epsilon} : \mathbb{C}[W_{\mathfrak{s}^\vee,\phi_1,v,q\epsilon}, \kappa_{\phi_1,v,q\epsilon}] \to \mathbb{C}[W_{\mathfrak{s}^\vee,\gamma(\phi_1),v,q\epsilon}, \kappa_{\gamma\phi_1,v,q\epsilon}].$$

Let us recall its construction. There is a G_{ϕ_1}-equivariant local system $\pi_*(\widetilde{q\mathcal{E}})$ on $(G_{\phi_1})_{\mathrm{RS}}$, an analogue of K and K^*. It satisfies

$$\mathbb{C}[W_{\mathfrak{s}^\vee,\phi_1,v,q\epsilon}, \kappa_{\phi_1,v,q\epsilon}] \cong \mathrm{End}_{\mathcal{D}(G_{\phi_1})_{\mathrm{RS}}}(\pi_*(\widetilde{q\mathcal{E}})). \tag{81}$$

Choosing a lift $n_\gamma \in N_{G_{\phi_1}}(M)$ of γ and following the proof of [AMS1, Lemma 5.4], we find an isomorphism

$$qb_\gamma : \pi_*(\widetilde{q\mathcal{E}}) \to \pi_*\big(\widetilde{\mathrm{Ad}(n_\gamma)^* q\mathcal{E}}\big). \tag{82}$$

Then $\psi_{\gamma,\phi_1,v,q\epsilon}$ is conjugation with qb_γ.

In this context [AMS1, Lemma 5.4] says that there are canonical elements $qb_w \in \mathrm{End}_{\mathcal{D}(G_{\phi_1})_{\mathrm{RS}}}(\pi_*(\widetilde{q\mathcal{E}}))$ ($w \in W_{\mathfrak{s}^\vee}^\circ$) which via (81) become a basis of $\mathbb{C}[W_{\mathfrak{s}^\vee}^\circ]$. Since $W_{\mathfrak{s}^\vee}^\circ$ is normal in $W_{\mathfrak{s}^\vee}$, $\psi_{\gamma,\phi_1,v,q\epsilon}$ stabilizes the set $\{qb_w : W_{\mathfrak{s}^\vee}^\circ\}$. Moreover $\gamma \in \mathfrak{R}_{\mathfrak{s}^\vee}$, so $\psi_{\gamma,\phi_1,v,q\epsilon}$ permutes the set of simple reflections in $W_{\mathfrak{s}^\vee}^\circ$.

From Proposition 2.1 and (28) we observe that the parameter functions λ and λ^* are $W_{\mathfrak{s}^\vee}$-invariant. Hence the map $N_{s_\alpha} \mapsto N_{\gamma s_\alpha \gamma^{-1}}$ extends uniquely to an automorphism of the Iwahori–Hecke algebra $\mathcal{H}(W_{\mathcal{E}}^\circ, \vec{\mathbf{z}}^{2\lambda})$ which fixes $\vec{\mathbf{z}}$.

Now we have canonical group actions of $\mathfrak{R}_{\mathfrak{s}^\vee}$ on the algebras

$$\mathcal{O}(X_{\mathrm{nr}}({}^L\mathcal{L}) \times (\mathbb{C}^\times)^d) = \mathbb{C}[X^*(X_{\mathrm{nr}}({}^L\mathcal{L}))] \otimes \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}]$$

and $\mathcal{H}(W_{\mathcal{E}}^\circ, \vec{\mathbf{z}}^{2\lambda})$, and as vector spaces

$$\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) = \mathcal{O}(X_{\mathrm{nr}}({}^L\mathcal{L}) \times (\mathbb{C}^\times)^d) \otimes \mathcal{H}(W_{\mathcal{E}}^\circ, \vec{\mathbf{z}}^{2\lambda}).$$

The relation involving $\theta_x N_{s_\alpha} - N_{s_\alpha} \theta_{s_\alpha(x)}$ in Proposition 2.2 is also preserved by γ, because $x(\gamma(1)) = s_\alpha(x)(\gamma(1))$. So $\mathfrak{R}_{\mathfrak{s}^\vee}$ acts canonically on $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$ by algebra automorphisms.

(b) The same construction as in the proof of Proposition 2.2 yields an algebra

$$\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}, \kappa_{\mathfrak{s}^\vee}], \tag{83}$$

in which the action of $\mathfrak{R}_{\mathfrak{s}^\vee}$ on $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$ has become an inner automorphism. This works for any 2-cocycle $\kappa_{\mathfrak{s}^\vee}$. It only remains to pick it in a good way, such that $\kappa_{\mathfrak{s}^\vee}|_{(W_{\mathfrak{s}^\vee,\phi,v,q\epsilon})^2}$ equals $\kappa_{\mathfrak{s}^\vee,\phi,v,q\epsilon}$. For this we, again, use the maps qb_γ from (82). The cuspidal local system $\mathrm{Ad}(n_\gamma)^* q\mathcal{E}$ does not depend on the choice of n_γ, because $q\mathcal{E}$ is M-equivariant. Furthermore qb_γ is unique up to scalars, so

$$qb_\gamma \cdot qb_{\gamma'} = \lambda_{\gamma,\gamma'} qb_{\gamma\gamma'} \text{ for a unique } \lambda_{\gamma,\gamma'} \in \mathbb{C}^\times .$$

We define $\kappa_{\mathfrak{s}^\vee}$ by $\kappa_{\mathfrak{s}^\vee}(\gamma,\gamma') = \lambda_{\gamma,\gamma'}$. This is a slight generalization of the construction in Section 1 and in [AMS1, Lemma 5.4]. As over there,

$$\begin{array}{lll} \mathrm{End}_{\mathcal{D}\mathrm{Lie}(J)_{\mathrm{RS}}}\big((\mathrm{pr}_1)_! q\dot{\mathcal{E}}\big) & \cong & \mathbb{C}[W_{\mathfrak{s}^\vee}, \kappa_{\mathfrak{s}^\vee}], \\ \mathrm{End}^+_{\mathcal{D}\mathrm{Lie}(J)_{\mathrm{RS}}}\big((\mathrm{pr}_1)_! q\dot{\mathcal{E}}\big) & \cong & \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}, \kappa_{\mathfrak{s}^\vee}]. \end{array}$$

As the J-equivariant sheaf $(\mathrm{pr}_1)_! q\dot{\mathcal{E}}$ on $\mathrm{Lie}(J)_{\mathrm{RS}}$ contains the G_ϕ-equivariant sheaf $(\mathrm{pr}_1)_! q\dot{\mathcal{E}}$ on $\mathrm{Lie}(G_\phi)_{\mathrm{RS}}$,

$$\kappa_{\mathfrak{s}^\vee} : (W_{\mathfrak{s}^\vee})^2 \to (W_{\mathfrak{s}^\vee}/W^\circ_{\mathfrak{s}^\vee})^2 = \mathfrak{R}^2_{\mathfrak{s}^\vee} \to \mathbb{C}^\times$$

extends $\kappa_{\mathfrak{s}^\vee,\phi,v,q\epsilon} : (W_{\mathfrak{s}^\vee,\phi,v,q\epsilon})^2 \to \mathbb{C}^\times$, for every $(\phi|_{\mathbf{W}_F}, v, q\epsilon) \in \mathfrak{s}^\vee_{\mathcal{L}}$. For $\phi = \phi_1$ this means that

$$\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[W_{\mathfrak{s}^\vee,\phi_1,v,q\epsilon}, \natural_{\mathfrak{s}^\vee,\phi_1,v,q\epsilon}].$$

is canonically embedded in (83). $\square$

The algebra from Proposition 3.12.b is attached to $\mathfrak{s}^\vee$ and the basepoint ϕ_1 of $\mathfrak{s}^\vee_{\mathcal{L}}$. To remove the dependence on the basepoint, we reinterpret $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$. Recall that $W_{\mathfrak{s}^\vee}$ acts naturally on $\mathfrak{s}^\vee_{\mathcal{L}}$ (which is diffeomorphic to $T_{\mathfrak{s}^\vee}$). In view of Lemma 3.10 every $\alpha \in R(J^\circ, T)$ is well-defined as a function on $\mathfrak{s}^\vee_{\mathcal{L}}$, that is, independent of ϕ_1. In the same way as in Proposition 2.2, we can define an algebra structure on

$$\mathcal{O}(\mathfrak{s}^\vee_{\mathcal{L}}) \otimes \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}] \otimes \mathbb{C}[W^\circ_{\mathfrak{s}^\vee}].$$

It becomes an algebra $\mathcal{H}(\mathfrak{s}^\vee_{\mathcal{L}}, W^\circ_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$ which is isomorphic to $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$, but only via the choice of a basepoint of $\mathfrak{s}^\vee_{\mathcal{L}}$. In Proposition 3.12.a we showed that $\mathfrak{R}_{\mathfrak{s}^\vee}$ acts naturally on $\mathcal{H}(\mathfrak{s}^\vee_{\mathcal{L}}, W^\circ_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$. Applying Proposition 3.12.b, we obtain an algebra

$$\mathcal{H}(\mathfrak{s}^\vee_{\mathcal{L}}, W^\circ_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathrm{End}^+_{\mathcal{D}\mathrm{Lie}(J)_{\mathrm{RS}}}\big((\mathrm{pr}_1)_! q\dot{\mathcal{E}}\big), \text{ where } J = Z^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{I}_F}). \tag{84}$$

Now we suppose that the almost direct factorization of J° is $W_{\mathfrak{s}^\vee}$-stable of the type (23). We obtain two algebras:

- $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$, the algebra (84) when $J_1 = J^\circ_{\mathrm{der}}$, with only one variable $\mathbf{z}$;
- $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$, the algebra (84) when (6) is the finest possible $W_{\mathfrak{s}^\vee}$-stable almost direct factorization of J°.

Lemma 3.13. *The algebras $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ and $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$ depend only on $\mathfrak{s}^\vee$, up to canonical isomorphisms.*

Proof. The above construction shows that $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ and $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$ are uniquely determined by $(\mathfrak{s}^\vee_{\mathcal{L}}, M, P)$. Up to $\mathcal{G}^\vee$-conjugation, this triple is completely determined by $\mathfrak{s}^\vee$. The normalizer of $\mathfrak{s}^\vee_{\mathcal{L}}$ is contained in J, and the pointwise stabilizer of $\mathfrak{s}^\vee_{\mathcal{L}}$ in J is just M. Given $\mathfrak{s}^\vee_{\mathcal{L}}$ and M, [AMS2, Lemma 1.1] shows that all possible choices for P are conjugate by unique elements of $N_{J^\circ}(M^\circ)/M^\circ$. Thus all possible $(\mathfrak{s}'^\vee_{\mathcal{L}}, M', P')$ underlying $\mathfrak{s}^\vee$ are conjugate to $(\mathfrak{s}^\vee_{\mathcal{L}}, M, P)$ in a canonical way. Any element of $\mathcal{G}^\vee_{\mathrm{sc}}$ which realizes such a conjugation provides a canonical isomorphism between $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ (respectively $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$) and its version based on $(\mathfrak{s}'^\vee_{\mathcal{L}}, M', P')$. $\square$

Example 3.14. Suppose that (ϕ, ρ) is itself cuspidal, so $\mathcal{L}^\vee = \mathcal{G}^\vee$ and $q\epsilon = \rho$. Then $J^\circ = M^\circ$, v is distinguished in that group, $T = 1$ and $R(J^\circ, T)$ is empty. Furthermore $W_{\mathfrak{s}^\vee} = 1$ because $N_{\mathcal{G}^\vee}(\mathcal{L}^\vee \rtimes \mathbf{W}_F)/\mathcal{L}^\vee = 1$. Consequently

$$\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z}) = \mathcal{O}(T_{\mathfrak{s}^\vee}) \otimes \mathbb{C}[\mathbf{z}, \mathbf{z}^{-1}] \quad \text{and} \quad \mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) = \mathcal{O}(T_{\mathfrak{s}^\vee}) \otimes \mathbb{C}[\mathbf{z}_1, \mathbf{z}_1^{-1}, \ldots, \mathbf{z}_d, \mathbf{z}_d^{-1}],$$

where d is the number of simple factors of J°_{der}.

For (ϕ, ρ) as in (71), let $\bar{M}(\phi, \rho, \vec{z})$ be the irreducible $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$-module obtained from $M(\phi, \rho, \log \vec{z}) \in \mathrm{Irr}(\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}}))$ via Theorems 2.5 and 2.9. Up to $\mathcal{G}^\vee$-conjugation, every element of $\Phi_e(\mathcal{G}^\vee)^{\mathfrak{s}^\vee}$ is of the form described in (71), so this definition extends naturally to all possible (ϕ, ρ). Similarly we define $\bar{E}(\phi, \rho, \vec{z})$ as the "standard" $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$-module obtained from $E(\phi, \rho, \log \vec{z}) \in \mathrm{Mod}(\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}}))$ via Theorems 2.5 and 2.9.

We formulate the next result only for $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$, but there is also a version for $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$. In view of (29), the latter can be obtained by assuming that all z_j are equal.

Theorem 3.15. (*a*) *For every $\vec{z} \in \mathbb{R}^d_{>0}$ there exists a canonical bijection*

$$\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee} \to \mathrm{Irr}_{\vec{z}}(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})) : \quad (\phi, \rho) \mapsto \bar{M}(\phi, \rho, \vec{z}).$$

(*b*) *Both $\bar{M}(\phi, \rho, \vec{z})$ and $\bar{E}(\phi, \rho, \vec{z})$ admit the central character $W_{\mathfrak{s}^\vee}(\tilde{\phi}|_{\mathbf{W}_F}, v, q\epsilon) \in \Phi_e(\mathcal{L}(F))^{\mathfrak{s}^\vee_{\mathcal{L}}}/W_{\mathfrak{s}^\vee}$, where $\tilde{\phi}|_{\mathbf{I}_F} = \phi|_{\mathbf{I}_F}$ and $\tilde{\phi}(\mathrm{Frob}_F) = \phi(\mathrm{Frob}_F)\vec{\phi}\big(1, \left(\begin{smallmatrix} \vec{z} & 0 \\ 0 & \vec{z}^{-1} \end{smallmatrix}\right)\big)$ with $\vec{\phi}$ as in* (50).

(*c*) *Suppose that $\vec{z} \in \mathbb{R}^d_{\geq 1}$. Equivalent are:*
- *ϕ is bounded;*
- *$\bar{E}(\phi, \rho, \vec{z})$ is tempered;*
- *$\bar{M}(\phi, \rho, \vec{z})$ is tempered.*

(*d*) *Suppose that $\vec{z} \in \mathbb{R}^d_{>1}$. Then ϕ is discrete if and only if $\bar{M}(\phi, \rho, \vec{z})$ is essentially discrete series. In this case $\bar{E}(\phi, \rho, \vec{z}) = \bar{M}(\phi, \rho, \vec{z})$.*

Proof. (a) Let us fix the bounded part ϕ_b and consider only ϕ in $X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b$. We need to construct a bijection between such (ϕ, ρ) and the set of irreducible $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$-modules on which $\vec{\mathbf{z}}$ acts as $\vec{z}$ and with $\mathcal{O}(\mathfrak{s}^\vee_{\mathcal{L}})$-weights in

$$W_{\mathfrak{s}^\vee}(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b, v, q\epsilon) \subset \mathfrak{s}^\vee_{\mathcal{L}}.$$

We want to apply Theorem 2.5.a here, although $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$ and $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$ need not be of the form $\mathcal{H}(G, M, q\mathcal{E})$. To see that this is allowed, pick a basepoint ϕ_1 as in Proposition 3.9. Then $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$ becomes a twisted affine Hecke algebra associated to a root datum, parameters, a finite group and a 2-cocycle. For such an algebra the proof of Theorem 2.5 works, it does not matter that the parameters can be different and that $\mathfrak{R}_{\mathfrak{s}^\vee}$ need not fix the basepoint of $T_{\mathfrak{s}^\vee}$.

Consider the twisted affine Hecke algebra $\mathcal{H}(\mathfrak{s}^\vee, \phi_b)$ with as data the torus $\mathfrak{s}^\vee_{\mathcal{L}}$, roots $\{\alpha \in R(J^\circ, T) : s_\alpha(\phi_b) = \phi_b\}$, the finite group $W_{\mathfrak{s}^\vee, \phi_b, v, q\epsilon}$, parameters λ, λ^* as in (78) and (79) and 2-cocycle $\natural_{q\mathcal{E}}$. The upshot of Theorem 2.5.a is a canonical bijection between the above irreducible $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$-modules and the irreducible modules of $\mathcal{H}(\mathfrak{s}^\vee, \phi_b)$ with central character in $(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b, v, q\epsilon) \times \{\vec{z}\}$.

With respect to the new basepoint ϕ_b, $\mathcal{H}(\mathfrak{s}^\vee, \phi_b)$ becomes isomorphic to a twisted affine Hecke algebra of the form described in Proposition 2.2. Then we can apply Theorem 2.9 to it, which relates its modules to those over a twisted graded Hecke algebra. Again it does not matter that the parameters of the affine Hecke algebra can differ from those in Theorem 2.9, this result applies to all possible parameters. The parameters of the resulting graded Hecke algebra are given by (43) and (42). Comparing that with (78), (79) and (70), we see that that graded Hecke algebra is none other than $\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}})$.

Thus Theorem 2.5.a yields a bijection between the above set of irreducible modules and the irreducible $\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}})$-modules with central character in $\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}) \times$

$\{\log \vec{z}\}$. By Theorem 3.8 this last collection is canonically in bijection with

$$ {}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b|_{\mathbf{W}_F}, v, q\epsilon). \tag{85} $$

The resulting bijection between (85) and the subset of $\mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}))$ with the appropriate central character could depend on the choice of an element in the $W_{\mathfrak{s}^\vee}$-orbit of ϕ_b. Fortunately, the proof of Lemma 2.8 applies also in this setting, and it entails that the bijection does not depend on such choices. Now we combine all these bijections, for the various ϕ_b. This gives a canonical bijection between $\Phi_e(\mathcal{G}^\vee)^{\mathfrak{s}^\vee}$ and $\mathrm{Irr}_{\vec{z}}(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}))$.
(b) By [AMS2, (86)] $E(\phi, \rho, \log \vec{z})$ admits the central character $(W_{\mathfrak{s}^\vee,\phi_b}\sigma_0 \vec{\phi}\left(\begin{smallmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z}\end{smallmatrix}\right), \log \vec{z})$, where σ_0 is given by (72). Applying Theorems 2.9 and 2.5 produces the representation $\bar{E}(\phi, \rho, \vec{z})$, with the central character that sends Frob_F to $\phi(\mathrm{Frob}_F)\vec{\phi}\big(1, \left(\begin{smallmatrix} \vec{z} & 0 \\ 0 & \vec{z}^{-1}\end{smallmatrix}\right)\big)$. That is just $W_{\mathfrak{s}^\vee}(\tilde{\phi}|_{\mathbf{W}_F}, v, q\epsilon)$. This also applies to the quotient $\bar{M}(\phi, \rho, \vec{z})$ of $\bar{E}(\phi, \rho, \vec{z})$.
(c) and (d) These follow from Theorem 3.8 (parts b and c), Theorem 2.9.d and Proposition 2.7. □

We note that the $\tilde{\phi}$ from Theorem 3.15.b is essentially the infinitesimal character of (ϕ, ρ), at least when $z = q_F^{1/2}$. The bijection obtained in part (a) is compatible with parabolic induction in the same sense as Corollary 2.12. For reference, we formulate this precisely.

Lemma 3.16. (*a*) *There is a natural isomorphism of* $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$*-modules*

$$ \mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) \underset{\mathcal{H}(\mathfrak{s}_Q^\vee, \vec{\mathbf{z}})}{\otimes} \bar{E}^Q_{\phi,\rho^Q,\vec{z}} \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\mathcal{S}_\phi^Q}(\rho^Q, \rho) \otimes \bar{E}_{\phi,\rho,\vec{z}}, $$

where the sum runs over all $\rho \in \mathrm{Irr}\big(\mathcal{S}_\phi\big)$ *with* ${}^L\Psi^Q(\phi, \rho^Q) = {}^L\Psi(\phi, \rho)$.
(*b*) *The multiplicity of* $\bar{M}_{\phi,\rho,\vec{z}}$ *in* $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) \underset{\mathcal{H}(\mathfrak{s}_Q^\vee, \vec{\mathbf{z}})}{\otimes} \bar{E}^Q_{\phi,\rho^Q,\vec{z}}$ *is* $[\rho^Q : \rho]_{\mathcal{S}_\phi^Q}$. *It already appears that many times as a quotient of* $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) \underset{\mathcal{H}(\mathfrak{s}_Q^\vee, \vec{\mathbf{z}})}{\otimes} \bar{M}^Q_{\phi,\rho^Q,\vec{z}}$.

Proof. As observed after (73), the bijection in Theorem 3.8.a is compatible with parabolic induction in the sense of Corollary 2.12. The bijection in Theorem 3.15.a is obtained from Theorem 3.8 by means of the reduction Theorems 2.5 and 2.9. Since these reduction theorems respect parabolic induction, Corollary 2.12 remains valid in the setting of Theorem 3.8, and it gives the desired results. □

4. The relation with the stable Bernstein center

Let $\Phi({}^L\mathcal{G})$ be the collection of $\mathcal{G}^\vee$-orbits of L-parameters. Recently, inspired by [Vog], Haines has considered the stable Bernstein center in [Hai]. We will explore below the relation of the latter with the Bernstein components $\Phi_e({}^L\mathcal{G})^{\mathfrak{s}^\vee}$.

The notion of stable Bernstein center which we employ here naturally lives on the Galois side. In principle it should be related to stable distributions on $\mathcal{G}(F)$ [Hai, §5.5], but that connection is currently highly conjectural. Because of that, we will consider it for all inner twists of a given reductive connected p-adic group $\mathcal{G}(F)$ simultaneously. Let $\mathcal{G}^*(F)$ be a quasi-split F-group which is an inner twist of $\mathcal{G}(F)$. The equivalence classes of inner twists of $\mathcal{G}^*$ are parametrized by the Galois cohomology group $H^1(F, \mathcal{G}^*_{\rm ad})$. For every $\alpha \in H^1(F, \mathcal{G}^*_{\rm ad})$, we will denote by $\mathcal{G}_\alpha(F)$ an inner twist of $\mathcal{G}^*(F)$ which is parametrized by α. By construction

$$\Phi_e({}^L\mathcal{G}) = \bigsqcup_{\alpha \in H^1(F, \mathcal{G}^*_{\rm ad})} \Phi_e(\mathcal{G}_\alpha(F)).$$

Definition 4.1. The infinitesimal character of an L-parameter $\phi \in \Phi({}^L\mathcal{G})$ (or an enhanced L-parameter $(\phi, \rho) \in \Phi_e({}^L\mathcal{G})$) is the L-parameter $\lambda_\phi \colon \mathbf{W}_F \to \mathcal{G}^\vee \rtimes \mathbf{W}_F$ (trivial on $\mathrm{SL}_2(\mathbb{C})$) defined by

$$\lambda_\phi(w) := \phi(w, d_w), \quad \text{for all } w \in \mathbf{W}_F,$$

where $d_w = \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})$.

Remark. As noticed in [Hai, § 5], if ϕ is relevant $\mathcal{G}(F)$, it may occur that λ_ϕ is not relevant for $\mathcal{G}(F)$ anymore. In contrast, for every $\phi \in \Phi({}^L\mathcal{G})$, we have $\lambda_\phi \in \Phi({}^L\mathcal{G})$, since λ_ϕ is relevant for $\mathcal{G}^*(F)$.

Definition 4.2. An inertial infinitesimal datum for $\Phi({}^L\mathcal{G})$ is a pair $(\mathcal{L}^\vee, \mathfrak{i}_{\mathcal{L}^\vee})$, where $\mathcal{L}^\vee \subset \mathcal{G}^\vee$ is a $\mathbf{W}_F$-stable Levi subgroup and $\mathfrak{i}_{\mathcal{L}^\vee}$ is a $X_{\rm nr}({}^L\mathcal{L})$-orbit of a discrete Langlands parameter $\lambda \colon \mathbf{W}_F \to \mathcal{L}^\vee \rtimes \mathbf{W}_F$ (trivial on $\mathrm{SL}_2(\mathbb{C})$). Another such object is regarded as equivalent if the two are conjugate by an element of $\mathcal{G}^\vee$. The equivalence class is denoted $\mathfrak{i} = (\mathcal{L}^\vee, \mathfrak{i}_{\mathcal{L}^\vee})_{\mathcal{G}^\vee} = [\mathcal{L}^\vee, \lambda]_{\mathcal{G}^\vee}$.

The stable Bernstein center for ${}^L\mathcal{G}$ is the ring of $\mathcal{G}^\vee$-invariant regular functions on the union of the algebraic varieties $\mathfrak{i}$ (see [Hai, §5.3] for the precise meaning).

We will attach to each inertial equivalence class for $\Phi_e(\mathcal{G}(F))$ the $\mathcal{G}^\vee$-orbit of an inertial infinitesimal datum, as follows:

Definition 4.3. For every cuspidal inertial equivalence class
$\mathfrak{s}^\vee = (\mathcal{L}(F), X_{\rm nr}({}^L\mathcal{L}) \cdot (\varphi, \epsilon))_{\mathcal{G}^\vee}$ we set

$$\inf(\mathfrak{s}^\vee) := (\mathcal{L}^\vee_\varphi, X_{\rm nr}({}^L\mathcal{L}_\phi) \cdot \lambda_\varphi)_{\mathcal{G}^\vee},$$

where $\mathcal{L}^\vee_\varphi$ is a Levi subgroup of $\mathcal{L}^\vee$ which contains minimally $\varphi(\mathbf{W}_F)$.

We remark that if φ has nontrivial restriction to $\mathrm{SL}_2(\mathbb{C})$, then we may have $\mathcal{L}^\vee_\varphi \subsetneq \mathcal{L}^\vee$ and $X_{\rm nr}({}^L\mathcal{L}) \subsetneq X_{\rm nr}({}^L\mathcal{L}_\phi)$.

Let $\mathfrak{B}^\vee_{\rm st}$ denote the set of equivalence classes of inertial infinitesimal data for $\Phi_e({}^L\mathcal{G})$. For every $\mathfrak{i} = [\mathcal{L}^\vee_1, \lambda]_{\mathcal{G}^\vee} \in \mathfrak{B}^\vee_{\rm st}$ we set:

$$\Phi_e({}^L\mathcal{G})_{\mathfrak{i}} := \left\{ (\phi, \rho)_{\mathcal{G}^\vee} \in \Phi_e({}^L\mathcal{G}) \, : \, \lambda_\phi \in X_{\rm nr}({}^L\mathcal{L}_1) \cdot \lambda \right\}.$$

In this way, we obtain a partition of the set $\Phi_e({}^L\mathcal{G})$ (a "stable Bernstein decomposition"):

$$\Phi_e({}^L\mathcal{G}) = \bigsqcup\nolimits_{\mathfrak{i} \in \mathfrak{B}^\vee_{\rm st}} \Phi_e({}^L\mathcal{G})_{\mathfrak{i}}. \tag{86}$$

It is worth to observe that, in contrast with Section 3, the above definitions involve only the Langlands parameter $\phi \in \Phi({}^L\mathcal{G})$ and not the enhancement of ϕ. In particular (ϕ, ρ) and (ϕ, ρ') are always contained in the same $\Phi_e({}^L\mathcal{G})_{\mathfrak{i}}$. Consequently the decomposition (86) is coarser than the Bernstein decomposition of $\Phi_e({}^L\mathcal{G})$ from (65). To make this precise, we define

$$\mathfrak{B}^\vee({}^L\mathcal{G}) := \bigsqcup_{\alpha \in H^1(F, \mathcal{G}^*_{\rm ad})} \mathfrak{B}^\vee(\mathcal{G}_\alpha(F)).$$

Theorem 4.4. *For* $\mathfrak{i} \in \mathfrak{B}^\vee_{\mathrm{st}}$ *we write* $\mathfrak{B}^\vee_{\mathfrak{i}} := \{\mathfrak{s}^\vee \in \mathfrak{B}^\vee({}^L\mathcal{G}) : \inf(\mathfrak{s}^\vee) = \mathfrak{i}\}$. *Then*

$$\Phi_e({}^L\mathcal{G})_{\mathfrak{i}} = \bigsqcup\nolimits_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee_{\mathfrak{i}}} \Phi_e({}^L\mathcal{G})^{\mathfrak{s}^\vee}.$$

Proof. This follows from the fact that for any enhanced Langlands parameter (ϕ, ρ) in $\Phi_e({}^L\mathcal{G})$, the infinitesimal character λ_ϕ of ϕ coincides with the infinitesimal character λ_φ of its cuspidal support $(\varphi, q\epsilon)$ [AMS1, (108)]. □

This theorem implies that (86) is a partition of $\Phi_e({}^L\mathcal{G})$ in subsets which are both unions of Bernstein components and of L-packets, in the sense that one piece contains (ϕ, ρ) if and only if it contains (ϕ, ρ').

Combining Theorems 4.4 and 3.15, we obtain:

Corollary 4.5. *For every* $\mathfrak{i} \in \mathfrak{B}^\vee_{\mathrm{st}}$ *and every* $\vec{z} \in \mathbb{R}^d_{>0}$, *there is a canonical bijection*

$$\Phi_e({}^L\mathcal{G})_{\mathfrak{i}} \longleftrightarrow \bigsqcup\nolimits_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee_{\mathfrak{i}}} \mathrm{Irr}_{\vec{z}}(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})).$$

Remark. It is natural to expect that a certain compatibility should exist between the algebras $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$ when $\mathfrak{s}^\vee$ runs over the set $\mathfrak{B}^\vee_{\mathfrak{i}}$, for a fixed $\mathfrak{i}$. A naive guess would be that there exist "spectral transfer morphisms" (defined as generalizations to twisted affine Hecke algebras of the notion introduced by Opdam in the case of affine ones in [Opd2]) between the algebras $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$ for $\mathfrak{s}^\vee \in \mathfrak{B}^\vee_{\mathfrak{i}}$, the role of the lowest algebra being played by an algebra $\mathcal{H}(\mathfrak{s}^\vee_1, \vec{\mathbf{z}})$, with $\mathfrak{s}^\vee_1 = (\mathcal{L}_1, \mathfrak{s}^\vee_{\mathcal{L}_1})_{\mathcal{G}^\vee}$.

5. Examples

In this section we will work out some affine Hecke algebras attached to Bernstein components of Langlands parameters. In the examples that we consider the local Langlands correspondence is known, and it matches Bernstein components on the Galois side with Bernstein components on the p-adic side. We will compare the Hecke algebras associated to Bernstein components that correspond under the LLC.

5.1. Inner twists of $\mathrm{GL}_n(F)$.

Recall that F is a local non-archimedean field, and let q_F be the cardinality of its residue field. Let D be a division algebra with centre F and $\dim_F(D) = d^2$. Take $m \in N$ and consider $\mathcal{G}(F) = \mathrm{GL}_m(D)$. It is an inner form of $\mathrm{GL}_n(F)$ with $n = md$. In fact $\mathcal{G}(F)$ becomes an inner twist if we regard D, the Hasse invariant $h(D) \in \{z \in \mathbb{C}^\times : z^d = 1\}$ or the associated character χ_D of $Z(\mathrm{SL}_n(\mathbb{C}))$ as part of the data. Up to conjugacy every Levi subgroup of $\mathcal{G}(F)$ is of the form

$$\mathcal{L}(F) = \prod\nolimits_j \mathrm{GL}_{m_j}(D) \quad \text{with } \sum\nolimits_j m_j = m.$$

Let $(\phi = \bigoplus_j \phi_j, \rho = \otimes_j \rho_j) \in \Phi_{\mathrm{cusp}}(\mathcal{L}(F))$. In [AMS1, Example 6.9] we worked out the shape of cuspidal Langlands parameters (ϕ_j, ρ_j) for $\mathrm{GL}_{m_j}(D)$. Namely

- $\phi_j = \phi_j|_{\mathbf{W}_F} \otimes S_{d_j}$ where S_{d_j} is the irreducible d_j-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$ and $\phi_j|_{\mathbf{W}_F}$ is an irreducible representation of dimension $m_j d/d_j$. (This says that ϕ_j is discrete.)
- $\mathcal{S}_{\phi_j} = Z(\mathrm{SL}_{m_j d}(\mathbb{C}))$ and ρ_j is the character associated to $\mathrm{GL}_{m_j}(D)$, that is, $\rho_j(\exp(2\pi i k/(m_j d)) I_{m_j d}) = h(D)^k$. (So (ϕ_j, ρ_j) is relevant for $\mathrm{GL}_{m_j}(D)$.)
- lcm $(d, m_j d/d_j) = m_j d$, or equivalently $\gcd(d, m_j d/d_j) = d/d_j$. (This guarantees cuspidality.)

It is known that two irreducible representation ϕ_j and ϕ_k of $\mathbf{W}_F$ are isomorphic up to an unramified character, if and only if their restrictions to $\mathbf{I}_F$ are isomorphic. Hence we can adjust the indexing so that $\phi|_{\mathbf{I}_F} = \bigoplus_i \phi_i^{\oplus e_i}|_{\mathbf{I}_F}$. Because the restriction of each ϕ_i to $\mathbf{I}_F$ decomposes as sum of irreducible representations of $\mathbf{I}_F$ with multiplicity one, we find that $R(J^\circ, T) \cong \prod_i A_{e_i - 1}$. To determine the Hecke algebra of the associated Bernstein component $\mathfrak{s}^\vee$ of $\Phi_e(\mathcal{G}(F))$, we make a simplifying assumption: if $m_i = m_j$ and ϕ_i differs from ϕ_j by an unramified twist, then $\phi_i = \phi_j$.

We adjust the indexing so that

$$\mathcal{L}(F) = \prod\nolimits_i \mathrm{GL}_{m_i}(D)^{e_i}, \quad \phi = \bigoplus\nolimits_i \phi_i^{\oplus e_i}, \quad \rho = \bigotimes\nolimits_i \rho_i^{\otimes e_i},$$

where ϕ_i and ϕ_j are not inertially equivalent if $i \neq j$. Let $\mathfrak{s}_i^\vee$ be the Bernstein component of $\Phi_e(\mathrm{GL}_{m_i e_i}(D))$ determined by $(\phi_i^{\oplus e_i}, \rho_i^{\otimes e_i})$. Choose an isomorphism $M_{d e_i m_i}(\mathbb{C}) \cong M_{m_i d/d_i}(\mathbb{C}) \otimes M_{d_i e_i}(\mathbb{C})$ and let 1_m be the multiplicative unit of the matrix algebra $M_m(\mathbb{C})$. Then

$$\begin{aligned}
&G_\phi = Z_{\mathrm{SL}_n(\mathbb{C})}(\phi(\mathbf{W}_F)) \cong \mathrm{SL}_n(\mathbb{C}) \cap \prod\nolimits_i (1_{m_i d/d_i} \otimes \mathrm{GL}_{d_i e_i}(\mathbb{C})) = \mathrm{SL}_n(\mathbb{C}) \cap \prod\nolimits_i G_{\phi,i},\\
&M \cong \mathrm{SL}_n(\mathbb{C}) \cap \prod\nolimits_i (1_{m_i d/d_i} \otimes \mathrm{GL}_{d_i}(\mathbb{C})^{e_i}),\\
&T \cong \mathrm{SL}_n(\mathbb{C}) \cap \prod\nolimits_i (1_{m_i d/d_i} \otimes Z(\mathrm{GL}_{d_i}(\mathbb{C}))^{e_i}), \quad R(G_\phi, T) \cong \prod\nolimits_i A_{e_i - 1},\\
&T_{\phi_i} = \{\phi_i \otimes \chi_i \in \Phi(\mathrm{GL}_{m_i}(D)) : \chi_i \in X_{\mathrm{nr}}({}^L\mathrm{GL}_{m_i}(D))\},\\
&T_{\mathfrak{s}^\vee} = \prod\nolimits_i T_{\mathfrak{s}_i^\vee} = \prod\nolimits_i (T_{\phi_i})^{e_i}, \quad W_{\mathfrak{s}^\vee} = W_{\mathfrak{s}^\vee,\phi} \cong \prod\nolimits_i S_{e_i}.
\end{aligned}$$

Furthermore we can decompose $u_\phi = \prod_i (u_{\phi,i})$, where $u_{\phi,i}$ belongs to the unique distinguished unipotent class of $1_{m_i d/d_i} \otimes \mathrm{GL}_{d_i}(\mathbb{C})^{e_i}$. By [Lus2, 2.13] this implies $c(\alpha) = 2 d_i$ for all $\alpha \in R(G_{\phi,i} T, T)$. Then $\lambda(\alpha) = d_i$ on $R(G_{\phi,i} T, T)$, whereas λ^* does not occur. We conclude that

$$\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) = \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \vec{\mathbf{z}}) \cong \bigotimes\nolimits_i \mathcal{H}(\mathrm{GL}_{e_i d_i}(\mathbb{C}), \mathrm{GL}_{d_i}(\mathbb{C})^{e_i}, v_i, \rho_i^{\otimes e_i}, \mathbf{z}_i), \tag{87}$$

a tensor product of affine Hecke algebras of type GL_{e_i} with parameters $\mathbf{z}_i^{d_i}$. Because $T_{\phi,i}$ is the quotient of $X_{\mathrm{nr}}({}^L\mathrm{GL}_{m_i}(D))$ by a group of finite order, say $t(\phi_i)$, the most appropriate specialization of (87) is at $\mathbf{z}_i = q_F^{t(\phi_i)/2}$. Indeed this recovers the exact parameters found by Sécherre in [Sec1, Théorème 4.6], see (89).

Now we consider Hecke algebras on the p-adic side. By the local Langlands correspondence for $\mathrm{GL}_{m_i}(D)$ (see [HiSa, §11] and [ABPS2, §2]), (ϕ_i, ρ_i) is associated to a unique essentially square-integrable representation $\sigma_i \in \mathrm{Irr}(\mathrm{GL}_{m_i}(D))$. Moreover the condition $\mathrm{lcm}(d, m_i d/d_i) = m_i d$ guarantees that σ_i is supercuspidal, by [DKV, Théorème B.2.b]. (This is a formal consequence of the Jacquet–Langlands correspondence, so in view of [Bad] it also holds in positive characteristic.) Hence

$$(\phi_i^{\oplus e_i}, \rho_i^{\otimes e_i}) \in \Phi_{\mathrm{cusp}}(\mathrm{GL}_{m_i}(D)^{e_i}) \quad \text{corresponds to} \quad \sigma_i^{\otimes e_i} \in \mathrm{Irr}_{\mathrm{cusp}}(\mathrm{GL}_{m_i}(D)^{e_i}).$$

Let $\mathfrak{s}_i$ denote the inertial equivalence class for $\mathrm{GL}_{m_i e_i}(D)$ determined by $(\mathrm{GL}_{m_i}(D)^{e_i}, \sigma_i^{\otimes e_i})$. In [SeSt1, Théorème 5.23] a $\mathfrak{s}_i$-type (J_i, τ_i) was constructed. The Hecke algebra for (J_i, τ_i) was analysed in [Sec1, Théorème 4.6], Sécherre found an isomorphism

$$\mathcal{H}(\mathrm{GL}_{m_i e_i}(D), J_i, \tau_i) \cong \mathcal{H}(\mathrm{GL}_{e_i}, q_F^{f_i}), \tag{88}$$

where the right hand side denotes an affine Hecke algebra of type GL_{e_i} with parameter $q_F^{f_i}$ (for a suitable $f_i \in \mathbb{N}$ depending only on σ_i or ϕ_i, see below). From the explicit description in [Sec1, §4] one sees readily that the isomorphism (88) respects the natural Hilbert algebra structures on both sides.

Remark. Let $t(\sigma_i)$ denote the torsion number of σ_i, *i.e.,* the number of unramified characters χ_i of $\mathrm{GL}_{m_i}(D)$ such that $\chi_i \otimes \sigma_i \cong \sigma_i$. It can also be described as the number $t(\phi_i)$ of unramified twists $z_i \in X_{\mathrm{nr}}({}^L\mathrm{GL}_{m_i}(D))$ such that $z_i\phi_i \cong \phi_i$ in $\Phi_{\mathrm{cusp}}(\mathrm{GL}_{m_i}(D))$.

If $D = F$, then $f_i = t(\sigma_i)$. In general, $f_i = s(\sigma_i)\, t(\sigma_i)$, where $s(\sigma_i)$ is the reducibility number of σ_i, as defined in [SeSt2, Introduction] (see also [Sec2, Theorem 4.6]). The number $s(\sigma_i)$ coincides with the invariant introduced in [DKV, Théorème B.2.b] (as it follows for instance from [BHLS, Eqn. (1.1) and Definition 2.2]), itself equal to the integer d_i. Hence f_i admits the following description in terms of Langlands parameters:

$$f_i = s(\sigma_i)t(\sigma_i) = d_i t(\phi_i). \tag{89}$$

Write $\mathcal{M}(F) = \prod_i \mathrm{GL}_{m_i}(D)^{e_i}$, $\sigma = \bigotimes_i \sigma_i^{\otimes e_i}$ and let $\mathfrak{s}$ be the inertial equivalence class of $(\mathcal{M}(F), \sigma)$ for $\mathrm{GL}_m(D)$. In [SeSt2, Theorem C] a $\mathfrak{s}$-type (J, τ) was constructed, as a cover of the product of the types (J_i, τ_i) for $\mathfrak{s}_i$. Moreover it was shown that

$$\mathcal{H}(\mathrm{GL}_m(D), J, \tau) \cong \bigotimes\nolimits_i \mathcal{H}(\mathrm{GL}_{e_i}, q_F^{f_i}). \tag{90}$$

Since (88) was an isomorphism of Hilbert algebras, so is (90). Notice that the right hand side is also the specialization of $\mathcal{H}(\mathfrak{s}^\vee, \vec{z})$ at $\mathbf{z}_i = q_F^{f_i/2}$. Thus there are equivalences of categories

$$\mathrm{Rep}(\mathrm{GL}_m(D))^{\mathfrak{s}} \cong \mathrm{Mod}\Big(\bigotimes\nolimits_i \mathcal{H}(\mathrm{GL}_{e_i}, q_F^{f_i})\Big) \cong \mathrm{Mod}\Big(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})/\big(\{\mathbf{z}_i - q_F^{f_i/2}\}_i\big)\Big). \tag{91}$$

It was shown in [BaCi, §5.4] that, since these equivalences come from isomorphisms of Hilbert algebras, they preserve temperedness of representations. Then [ABPS4, Lemma 16.5] proves that (91) maps essentially square-integrable representations to essentially discrete series representations and conversely.

The torus underlying $\bigotimes_i \mathcal{H}(\mathrm{GL}_{e_i}, q_F^{f_i})$ is $T_{\mathfrak{s}} = [\mathcal{M}(F), \sigma]_{\mathcal{M}(F)}$, which by the LLC for $\mathrm{GL}_{m_i}(D)$ is naturally isomorphic to the torus $T_{\mathfrak{s}^\vee}$ underlying $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$. Then [ABPS3, Theorem 4.1] shows that, with the interpretation as in Lemma 3.13 (which highlights the tori in these affine Hecke algebras), the equivalences (91) become canonical. This means in essence that we use the local Langlands correspondence for supercuspidal representations as input. With Theorem 3.15 we obtain canonical bijections

$$\mathrm{Irr}(\mathrm{GL}_m(D))^{\mathfrak{s}} \longleftrightarrow \mathrm{Irr}\Big(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})/\big(\{\mathbf{z}_i - q_F^{f_i/2}\}_i\big)\Big) \longleftrightarrow \Phi_e(\mathrm{GL}_m(D))^{\mathfrak{s}^\vee}. \tag{92}$$

Proposition 5.1. *The union of the bijections* (92) *over all Bernstein components for* $\mathrm{GL}_m(D)$ *equals the local Langlands correspondence for* $\mathrm{GL}_m(D)$*.*

Proof. In [ABPS2, §2] the LLC for $\mathrm{GL}_m(D)$ was constructed by starting with irreducible essentially square-integrable representations of Levi subgroups, then applying parabolic induction and finally taking Langlands quotients. In the context of types and covers thereof, [BuKu1, Corollary 8.4] shows that the maps (91) commute with parabolic induction. They also commute with taking Langlands quotients, because for these groups every Langlands quotient is the unique irreducible quotient of a suitable representation.

Thus we have reduced the claim to the case of irreducible essentially square-integrable representations. From [DKV, §B.2] we see that $\mathrm{Rep}(\mathrm{GL}_m(D))^{\mathfrak s}$ only contains such representations if $m_1 e_1 = m$. We may just as well consider the group $\mathrm{GL}_{m_i e_i}(D)$, which we prefer because then we can stick to the above notation. All its irreducible essentially square-integrable representations are generalized Steinberg representations built from $T_{\mathfrak s_i}$. By construction the bijection (92) for $\mathrm{GL}_{m_i}(D)^{e_i}$ sends $T_{\mathfrak s_i}$ to $T_{\mathfrak s_i^\vee}$.

Let $\chi_i \in X_{\mathrm{nr}}(\mathrm{GL}_{m_i}(D))$, with Langlands parameter $t_i \in X_{\mathrm{nr}}({}^L \mathrm{GL}_{m_i}(D))$. The generalized Steinberg representation $\mathrm{St}(\sigma')$ based on $\sigma' = (\chi_i \sigma_i)^{\otimes e_i}$ is the irreducible essentially square-integrable subrepresentation of the parabolic induction of

$$\nu_i^{(1-e_i)/2} \chi_i \sigma_i \otimes \cdots \otimes \nu_i^{(e_i - 1)/2} \chi_i \sigma_i \tag{93}$$

to $\prod_i \mathrm{GL}_{m_i e_i}(D)$, where ν_i denotes the absolute value of reduced norm map for $\mathrm{GL}_{m_i}(D)$. There is a unique such subrepresentation by [DKV, Théorème B.2.b]. By definition [ABPS2, (12)] $\mathrm{St}(\sigma')$ has Langlands parameter $t_i \phi_i \otimes \pi_{e_i, \mathrm{SL}_2(\mathbb C)}$.

Now we plug $\mathrm{St}(\sigma')$ in (92) and we use the property discussed under (91). Thus we end up with an essentially discrete series representation of $\mathcal H(\mathfrak s^\vee, \vec{\mathbf z}) / \big(\{ \mathbf z_i - q_F^{f_i/2} \}_i \big)$. By Theorem 3.15 it corresponds to a discrete element of $\Phi_e(\mathrm{GL}_{m_i e_i}(D))^{\mathfrak s_i^\vee}$. Its enhancement ρ_i is uniquely determined by the requirement that it is relevant for $\mathrm{GL}_{m_i e_i}(D)$, so we can ignore that and focus on the L-parameter. The image of $\mathbf W_F$ under this L-parameter is contained in $\mathrm{GL}_{m_i}(D)^{e_i,\vee} = \mathrm{GL}_{m_i d}(\mathbb C)^{e_i}$, so it can only be discrete if it is of the form $\psi_i \otimes \pi_{e_i, \mathrm{SL}_2(\mathbb C)}$ for some irreducible $m_i d$-dimensional representation of $\mathbf W_F$. Since the cuspidal support of the enhanced L-parameter lies in $T_{\mathfrak s_i^\vee}$, ψ_i must be an unramified twist of ϕ_i. From (93) and the expression for the central character of $M(\psi_i \otimes \pi_{e_i, \mathrm{SL}_2(\mathbb C)}, \rho_i, z_i)$ given in Theorem 3.15.b we deduce that $\psi_i = t_i \phi_i$. Thus (92) agrees with the local Langlands correspondence for essentially square-integrable representations. □

5.2. **Inner twists of** $\mathrm{SL}_n(F)$.

This paragraph is largely based on [ABPS2, ABPS3]. We keep the notations from the previous paragraph. For any subgroup of $\mathrm{GL}_m(D)$, we indicate the subgroup of elements of reduced norm 1 by a $\sharp$. Thus

$$\mathcal G^\sharp(F) = \mathrm{GL}_m(D)^\sharp = \{ g \in \mathrm{GL}_m(D) : \mathrm{Nrd}(g) = 1 \} = \mathrm{SL}_m(D).$$

The inner twists of $\mathrm{GL}_n(F)$ are in bijection with the inner twists of $\mathrm{SL}_n(F)$, via $\mathrm{GL}_m(D) \leftrightarrow \mathrm{GL}_m(D)^\sharp = \mathrm{SL}_m(D)$. The L-parameters for $\mathrm{GL}_m(D)^\sharp$ are the same as for $\mathrm{GL}_m(D)$, only their image is considered in $\mathrm{PGL}_n(\mathbb C)$. In particular every discrete L-parameter

$$\phi^\sharp : \mathbf W_F \times \mathrm{SL}_2(\mathbb C) \to \mathrm{PGL}_n(\mathbb C)$$

lifts to an irreducible n-dimensional representation of $\mathbf W_F \times \mathrm{SL}_2(\mathbb C)$. The local Langlands correspondence for these groups was worked out in [HiSa, ABPS2]. It provides a bijection between the Bernstein components on both sides of the LLC, which will use implicitly as $\mathfrak s^\sharp \leftrightarrow \mathfrak s^{\sharp\vee}$.

Let $\phi = \otimes_i \phi_i^{\otimes e_i}$ be as before, and let $\phi^\sharp \in \Phi(\mathcal L^\sharp(F))$ be the obtained by composition with the projection $\mathrm{GL}_n(\mathbb C) \to \mathrm{PGL}_n(\mathbb C)$. Every Bernstein component contains

L-parameters of this form. There is a central extension

$$1 \to \mathcal{Z}_{\phi^\sharp} \to \mathcal{S}_{\phi^\sharp} \to \mathcal{R}_{\phi^\sharp} \to 1$$

where $\mathfrak{R}_{\phi^\sharp} = \pi_0(Z_{\mathrm{PGL}_n(\mathbb{C})}(\mathrm{im}\phi^\sharp))$ and

$$\mathcal{Z}_{\phi^\sharp} = Z(\mathrm{SL}_n(\mathbb{C}))/Z(\mathrm{SL}_n(\mathbb{C})) \cap Z_{\mathrm{SL}_n(\mathbb{C})}(\phi^\sharp)^\circ .$$

Let $\rho^\sharp$ be an enhancement of $\phi^\sharp$. The restriction $\rho = \rho^\sharp|_{\mathcal{Z}_{\phi^\sharp}}$ is an enhancement of ϕ, so as before we may assume that it has the form $\rho = \otimes_i \rho_i^{\otimes e_i}$. Cuspidality of $(\phi^\sharp, \rho^\sharp)$ depends only (ϕ, ρ), it holds whenever ρ_i is associated to the inner twist $\mathrm{GL}_{m_i}(D)$ of $\mathrm{GL}_n(F)$ via the Kottwitz isomorphism. We assume that this is the case, and that $(\phi^\sharp, \rho^\sharp) \in \Phi_{\mathrm{cusp}}(\mathcal{L}^\sharp(F))$. We note that $\mathcal{G}^\vee_{\mathrm{sc}}$ is the same for $\mathrm{GL}_m(D)$ and $\mathrm{SL}_m(D)$, and that ϕ and $\phi^\sharp$ have the same connected centralizer. Consequently

$$\begin{aligned} & G^\circ_{\phi^\sharp} = G^\circ_\phi, \quad G_{\phi^\sharp}/G^\circ_{\phi^\sharp} \cong \mathfrak{R}_{\phi^\sharp}, \quad M^\circ_{\phi^\sharp} = M^\circ_\phi, \\ & R(G^\circ_{\phi^\sharp}, T) = \prod\nolimits_i A_{e_i - 1}, \quad \lambda(\alpha) = d_i \ \forall \alpha \in R(G_{\phi,i}T, T) \subset R(G^\circ_{\phi^\sharp}, T). \end{aligned}$$

Let $\mathfrak{s}^{\sharp\vee}$ be the inertial equivalence class for $\Phi_e(\mathrm{GL}_m(D)^\sharp)$ determined by $(\phi^\sharp, \rho^\sharp)$. (In spite of the notation $\mathfrak{s}^\vee$ does not determine it uniquely.) Then

$$T_{\mathfrak{s}^{\sharp\vee}} = \big(\prod\nolimits_i T_{\phi_i}^{e_i}\big)/Z(\mathrm{GL}_n(\mathbb{C})), \quad W^\circ_{\mathfrak{s}^{\sharp\vee}} \cong \prod\nolimits_i S_{e_i}.$$

The cuspidal local system $q\mathcal{E}$ associated to $(\phi^\sharp, \rho^\sharp)$ satisfies

$$\mathfrak{R}_{q\mathcal{E}} \cong W_{\mathfrak{s}^{\sharp\vee}}/W^\circ_{\mathfrak{s}^{\sharp\vee}} = \mathfrak{R}_{\mathfrak{s}^{\sharp\vee}} \cong \mathfrak{R}_{\phi^\sharp}.$$

The algebra

$$\mathcal{H}(\mathcal{R}_{\mathfrak{s}^{\sharp\vee}}, \lambda, \vec{\mathbf{z}}) = \mathcal{H}(G^\circ_{\phi^\sharp}, M^\circ_{\phi^\sharp}, v, \rho, \vec{\mathbf{z}}) \tag{94}$$

is a subalgebra of $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \vec{\mathbf{z}})$, corresponding to the projection $T_{\mathfrak{s}^\vee} \to T_{\mathfrak{s}^{\sharp\vee}}$. It is contained in

$$\mathcal{H}(\mathfrak{s}^{\vee\sharp}, \vec{\mathbf{z}}) = \mathcal{H}(\mathcal{R}_{\mathfrak{s}^{\sharp\vee}}, \lambda, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\mathfrak{R}_{\phi^\sharp}, \natural_{\phi^\sharp}].$$

Here the twisted group algebra and the 2-cocycle $\natural_{\phi^\sharp} = \natural_{\mathfrak{s}^{\sharp\vee}}$ are given by

$$\mathbb{C}[\mathfrak{R}_{\phi^\sharp}, \natural_{\phi^\sharp}] = p_\rho \mathbb{C}[\mathcal{S}_{\phi^\sharp}],$$

while the action of $\mathfrak{R}_{\phi^\sharp}$ on (94) comes from its natural action on $\mathcal{R}_{\mathfrak{s}^{\sharp\vee}}$.

For better comparison with the p-adic side we also determine the graded Hecke algebras attached to $\mathfrak{s}^{\sharp\vee}$. Let $(\phi_b^\sharp, \rho^\sharp) \in \Phi_{\mathrm{cusp}}(\mathcal{L}^\sharp(F))$ be an unramified twist of $(\phi^\sharp, \rho^\sharp)$ which is bounded. Let $W_{\phi_b^\sharp}$ be the stabilizer of $\phi_b^\sharp$ in $W_{\mathfrak{s}^{\sharp\vee}}$. Then $W^\circ_{\phi_b^\sharp} = W(G^\circ_{\phi_b^\sharp}, T)$ is the subgroup of $W_{\phi_b^\sharp} \cap W^\circ_{\mathfrak{s}^{\sharp\vee}}$ generated by the reflections it contains. The parabolic subgroup of $G^\circ_{\phi_b^\sharp}$ generated by $M^\circ_{\phi_b^\sharp}$ and upper triangular matrices determines a group $\mathfrak{R}_{\phi_b^{\sharp\vee}}$ such that

$$W_{\phi_b^\sharp} = W^\circ_{\phi_b^\sharp} \rtimes \mathfrak{R}_{\phi_b^\sharp}.$$

The 2-cocycle $\natural_{\phi_b^\sharp}$ on $W_{\phi_b^\sharp}$ is the restriction of $\natural_{\mathfrak{s}^{\sharp\vee}} : W^2_{\mathfrak{s}^{\sharp\vee}} \to \mathbb{C}^\times$. The root system $R_{\phi_b^\sharp}$ is again a product of systems of type A, namely $\prod_j A_{\epsilon_j - 1}$ if $\phi_b^\sharp = \otimes_j \phi_j^{\epsilon_j}$. Then $W^\circ_{\phi_b^\sharp} \cong \prod_j S_{\epsilon_j}$ and

$$\mathfrak{t}_{\mathfrak{s}^{\sharp\vee}} = \mathrm{Lie}(T_{\mathfrak{s}^{\sharp\vee}}) = \big(\sum\nolimits_i \mathrm{Lie}(T_{\phi_i}^{e_i})\big)/Z(\mathfrak{gl}_n(\mathbb{C})).$$

It follows that

$$\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}}) \cong \mathbb{H}(\mathfrak{t}_{\mathfrak{s}^\sharp\vee}, W_{\phi_b^\sharp}, \vec{\mathbf{r}}, \natural_{\phi_b^\sharp}) \cong \mathbb{H}(\mathfrak{t}_{\mathfrak{s}^\sharp\vee}, W_{\phi_b^\sharp}, \vec{\mathbf{r}}) \rtimes \mathbb{C}[\mathfrak{R}_{\phi_b^\sharp}, \natural_{\phi_b^\sharp}]. \tag{95}$$

The Hecke algebras for Bernstein components of $\mathrm{SL}_m(D)$ were computed in [ABPS3]. They are substantially more complicated than their counterparts for $\mathrm{GL}_m(D)$, and in particular do not match entirely with the above affine Hecke algebras for Langlands parameters. To describe them, we need some notations. Let $\mathcal{P}$ be a parabolic subgroup of $GL_m(D)$, with Levi factor $\mathcal{M}$. Consider the inertial equivalence classes $\mathfrak{s}_\mathcal{M} = [\mathcal{M}, \sigma]_\mathcal{M}$ and $\mathfrak{s} = [\mathcal{M}, \sigma]_{\mathrm{GL}_m(D)}$. Recall from (90) that $\mathcal{H}(\mathrm{GL}_m(D))^\mathfrak{s}$ is Morita equivalent with

$$\mathcal{H}(\mathcal{R}_\mathfrak{s}, \lambda, q_\mathfrak{s}) = \bigotimes\nolimits_i \mathcal{H}(\mathrm{GL}_{e_i}, q_F^{f_i}).$$

We need the groups

$$X^\mathcal{M}(\mathfrak{s}) = \big\{ \gamma \in \mathrm{Irr}\big(\mathcal{M}/\mathcal{M}^\sharp Z(\mathrm{GL}_m(D))\big) : \gamma \otimes \sigma \in \mathfrak{s}_\mathcal{M} \big\},$$

$$X^{\mathrm{GL}_m(D)}(\mathfrak{s}) = \big\{ \gamma \in \mathrm{Irr}\big(\mathrm{GL}_m(D)/\mathrm{GL}_m(D)^\sharp Z(\mathrm{GL}_m(D))\big) : \gamma \otimes I_\mathcal{P}^{\mathrm{GL}_m(D)}(\sigma) \in \mathfrak{s} \big\},$$

$$W_\mathfrak{s}^\sharp = \big\{ w \in N_{\mathrm{GL}_m(D)}(\mathcal{M})/(\mathcal{M}) : \exists \gamma \in \mathrm{Irr}\big(\mathcal{M}/\mathcal{M}^\sharp Z(\mathrm{GL}_m(D))\big) : w(\gamma \otimes \sigma) \in \mathfrak{s}_\mathcal{M} \big\}.$$

By [ABPS3, Lemma 2.3] $W_\mathfrak{s}^\sharp = W_\mathfrak{s} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp$ for a suitable subgroup $\mathfrak{R}_\mathfrak{s}^\sharp$, and by [ABPS3, Lemma 2.4] $X^{\mathrm{GL}_m(D)}(\mathfrak{s})/X^\mathcal{M}(\mathfrak{s}) \cong \mathfrak{R}_\mathfrak{s}^\sharp$. The group $X^{\mathrm{GL}_m(D)}(\mathfrak{s})$ acts naturally on $T_\mathfrak{s} \rtimes W_\mathfrak{s}$.

Let $\sigma^\sharp$ be an irreducible constituent of $\sigma|_{\mathcal{M}^\sharp}$. Every inertial equivalence class for $\mathrm{SL}_m(D) = \mathrm{GL}_m(D)^\sharp$ is of the form $\mathfrak{s}^\sharp = [\mathcal{M}^\sharp, \sigma^\sharp]_{\mathrm{GL}_m(D)^\sharp}$. By [ABPS3, Theorem 1] there exists a finite dimensional projective representation V_μ of $X^{\mathrm{GL}_m}\mathfrak{s})$ such that $\mathcal{H}(\mathrm{GL}_m(D)^\sharp)^{\mathfrak{s}^\sharp}$ is Morita equivalent with one direct summand of

$$\big(\mathcal{H}(\mathcal{R}_\mathfrak{s}, \lambda, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^\mathcal{M}(\mathfrak{s}) X_{\mathrm{nr}}(\mathcal{M}/\mathcal{M}^\sharp)} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp. \tag{96}$$

The other direct summands correspond to different constituents of $\sigma|_{\mathcal{M}^\sharp}$. In (96) the group

$$X_{\mathrm{nr}}(\mathcal{M}/\mathcal{M}^\sharp) = \{\chi \in X_{\mathrm{nr}}(\mathcal{M}) : \mathcal{M}^\sharp \subset \ker \chi\}$$

acts only via translations of $T_\mathfrak{s}$. We denote the quotient torus $T_\mathfrak{s}/X_{\mathrm{nr}}(\mathcal{M}/\mathcal{M}^\sharp)$ by $T_\mathfrak{s}^\sharp$ and its Lie algebra by $\mathfrak{t}_\mathfrak{s}^\sharp$.

From now on we will be more sketchy. The below can be made precise, but for that one would have to delve into some of the technicalities of [ABPS3], which are not so relevant for this paper. Although it is not so easy to write down all direct summands of (96) explicitly, we can say that they look like

$$\big(\mathcal{H}(X^*(T_\mathfrak{s}^\sharp), R_\mathfrak{s}, X_*(T_\mathfrak{s}^\sharp), R_\mathfrak{s}^\vee, \lambda, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_{\mu^\sharp})\big)^{X^\mathcal{M}(\mathfrak{s}, \sigma^\sharp)} \rtimes \mathfrak{R}_{\mathfrak{s}^\sharp} \tag{97}$$

for suitable $X^\mathcal{M}(\mathfrak{s}, \sigma^\sharp) \subset X^\mathcal{M}(\mathfrak{s})$ and $V_{\mu^\sharp} \subset V_\mu$. (From the below argument for graded Hecke algebras one see approximately how (97) arises from (96).) This algebra need not be Morita equivalent to a twisted affine Hecke algebra as studied in this paper. The problem comes from the simultaneous action of $X^\mathcal{M}(\mathfrak{s}, \sigma^\sharp)$ on $T_\mathfrak{s}^\sharp$ and $V_{\mu^\sharp}$: if that is complicated, it prevents (97) from being Morita equivalent to a similar algebra without $\mathrm{End}_\mathbb{C}(V_{\mu^\sharp})$. If we consider (97) as a kind of algebra bundle over $T_\mathfrak{s}^\sharp$, then these remarks mean that $V_{\mu^\sharp}$ could introduce some extra twists in this bundle, which take the algebra outside the scope of this paper. Examples can be constructed by combining the ideas in [ABPS3, Examples 5.2 and 5.5].

That being said, the other data involved in (97) are as desired. It was checked in [ABPS5, Lemma 5.5] that:

(i) The underlying torus $T_{\mathfrak{s}^\sharp} = T_{\mathfrak{s}}^\sharp / X^{\mathcal{M}}(\mathfrak{s}, \sigma^\sharp)$ is naturally isomorphic to $T_{\mathfrak{s}^{\sharp\vee}} = \Phi_e(\mathcal{M}^\sharp)^{[\mathcal{M}^\sharp,\sigma^\sharp]_{\mathcal{M}^\sharp}}$.
(ii) $W_{\mathfrak{s}} \rtimes \mathfrak{R}_{\mathfrak{s}^\sharp} = W_{\mathfrak{s}^\sharp}$ is isomorphic to $W_{\mathfrak{s}^{\sharp\vee}} = W_{\mathfrak{s}^\vee} \rtimes \mathfrak{R}_{\mathfrak{s}^{\sharp\vee}}$.
(iii) The space of irreducible representations of (97) is isomorphic to a twisted extended quotient

$$(T_{\mathfrak{s}^\sharp} // W_{\mathfrak{s}^\sharp})_{\kappa_{\sigma^\sharp}} \cong (T_{\mathfrak{s}^{\sharp\vee}} // W_{\mathfrak{s}^{\sharp\vee}})_{\kappa_{\sigma^\sharp}},$$

and the 2-cocycle $\kappa_{\sigma^\sharp}$ of $W_{\mathfrak{s}^\sharp}$ is equivalent to the 2-cocycle $\natural_{\mathfrak{s}^{\sharp\vee}}$ of $W_{\mathfrak{s}^{\sharp\vee}}$.

Let us also discuss the graded Hecke algebras which can be derived from (96) and (97). The algebra $\mathcal{O}(T_{\mathfrak{s}}^\sharp)^{X^{\mathcal{M}}(\mathfrak{s}) W_{\mathfrak{s}}}$ is naturally contained in the centre of (96). This entails that we can localize at suitable subsets of $T_{\mathfrak{s}}^\sharp / W_{\mathfrak{s}}^\sharp X^{\mathcal{M}}(\mathfrak{s})$. Fix $t \in (T_{\mathfrak{s}}^\sharp)_{\mathrm{un}}$. By localization at a small neighborhood of U of $W_{\mathfrak{s}}^\sharp X^{\mathcal{M}}(\mathfrak{s}) t (T_{\mathfrak{s}}^\sharp)_{\mathrm{rs}}$, we can effectively replace $X^{\mathcal{M}}(\mathfrak{s})$ by the stabilizer of $X^{\mathcal{M}}(\mathfrak{s})_t$, and $\mathfrak{R}_{\mathfrak{s}}^\sharp$ by the stabilizer $\mathfrak{R}_{\mathfrak{s}}^\sharp(t)$ of $W_{\mathfrak{s}} X^{\mathcal{M}}(\mathfrak{s}) t$. Then (96) is transformed into the algebra

$$C_{an}(U)^{X^{\mathcal{M}}(\mathfrak{s}) W_{\mathfrak{s}}^\sharp} \underset{\mathcal{O}(T_{\mathfrak{s}}^\sharp)^{X^{\mathcal{M}}(\mathfrak{s}) W_{\mathfrak{s}}^\sharp}}{\otimes} \big(\mathcal{H}(\mathcal{R}_{\mathfrak{s}}^\sharp, \lambda, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) \big)^{X^{\mathcal{M}}(\mathfrak{s})_t} \rtimes \mathfrak{R}_{\mathfrak{s}^\sharp}(t) \tag{98}$$

where $\mathcal{R}_{\mathfrak{s}}^\sharp = (X^*(T_{\mathfrak{s}}^\sharp), R_{\mathfrak{s}}, X_*(T_{\mathfrak{s}}^\sharp), R_{\mathfrak{s}}^\vee)$. But $X^{\mathcal{M}}(\mathfrak{s})$ acts by translations on $T_{\mathfrak{s}}^\sharp$, so $X^{\mathcal{M}}(\mathfrak{s})_t$ consists of all the elements that fix $T_{\mathfrak{s}}^\sharp$ entirely. From the description of the actions on (96) in [ABPS3, Lemma 4.11] we see that $X^{\mathcal{M}}(\mathfrak{s})_t$ acts only on $\mathrm{End}_{\mathbb{C}}(V_\mu)$. Then

$$\mathrm{End}_{\mathbb{C}}(V_\mu)^{X^{\mathcal{M}}(\mathfrak{s})_t} = \mathrm{End}_{X^{\mathcal{M}}(\mathfrak{s})_t}(V_\mu) \cong \bigoplus\nolimits_{\mu^\sharp} \mathrm{End}_{\mathbb{C}}(V_{\mu^\sharp}) \tag{99}$$

is a finite dimensional semisimple algebra. The direct summands of (96) and of (98) are in bijection with the $\mathfrak{R}_{\mathfrak{s}}^\sharp(t)$-orbits on the set of direct summands of (99). That holds for any $t \in (T_{\mathfrak{s}}^\sharp)_{\mathrm{un}}$, in particular for s ome t with $\mathfrak{R}_{\mathfrak{s}}^\sharp(t) = 1$, so in fact the direct summands $\mathrm{End}_{\mathbb{C}}(V_{\mu^\sharp})$ of (99) parametrize the direct summands of (96) and of (98). Thus (98) is a direct sum of algebras

$$C_{an}(U)^{X^{\mathcal{M}}(\mathfrak{s}) W_{\mathfrak{s}}^\sharp} \underset{\mathcal{O}(T_{\mathfrak{s}}^\sharp)^{X^{\mathcal{M}}(\mathfrak{s}) W_{\mathfrak{s}}^\sharp}}{\otimes} \big(\mathcal{H}(\mathcal{R}_{\mathfrak{s}}^\sharp, \lambda, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\mu^\sharp}) \big) \rtimes \mathfrak{R}_{\mathfrak{s}^\sharp}(t). \tag{100}$$

Here $(\mu^\sharp, V_{\mu^\sharp})$ is a projective representation of $\mathfrak{R}_{\mathfrak{s}^\sharp}(t)$. In such situations there is a Morita equivalent algebra embedding

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\sharp}(t), \natural] & \to & \mathrm{End}_{\mathbb{C}}(V_{\mu^\sharp}) \rtimes \mathfrak{R}_{\mathfrak{s}^\sharp}(t) \\ r & \mapsto & \mu^\sharp(r)^{-1} r, \end{array}$$

for a suitable 2-cocycle $\natural$. Via this method (100) is Morita equivalent with

$$C_{an}(U)^{X^{\mathcal{M}}(\mathfrak{s}) W_{\mathfrak{s}}^\sharp} \underset{\mathcal{O}(T_{\mathfrak{s}}^\sharp)^{X^{\mathcal{M}}(\mathfrak{s}) W_{\mathfrak{s}}^\sharp}}{\otimes} \mathcal{H}(\mathcal{R}_{\mathfrak{s}}^\sharp, \lambda, q_{\mathfrak{s}}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\sharp}(t), \natural]. \tag{101}$$

From the property (iii) of the algebra (97) we see that $\natural$ has to be the restriction of $\natural_{\mathfrak{s}^{\sharp\vee}}$ to $\mathfrak{R}_{\mathfrak{s}^\sharp}(t)^2$. By Theorems 2.5.a and 2.9.a the algebra (101) is Morita equivalent with

$$C_{an}(U)^{X^{\mathcal{M}}(\mathfrak{s})W_{\mathfrak{s}}^\sharp} \underset{\mathcal{O}(\mathfrak{t}_{\mathfrak{s}}^\sharp)^{X^{\mathcal{M}}(\mathfrak{s})W_{\mathfrak{s}}^\sharp}}{\otimes} \mathbb{H}(\mathfrak{t}_{\mathfrak{s}}^\sharp, W(R_{\mathfrak{s}})_t, q_{\mathfrak{s}}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\sharp}(t), \natural_{\mathfrak{s}^{\sharp\vee}}]. \tag{102}$$

Hence the equivalence between $\mathrm{Rep}(\mathrm{SL}_m(D))^{\mathfrak{s}^\sharp} \cong \mathrm{Mod}\big(\mathcal{H}(\mathrm{GL}_m(D)^\sharp)^{\mathfrak{s}^\sharp}\big)$ and the module category of (97) restricts to an equivalence between

$$\begin{aligned} &\mathrm{Mod}_{f,W_{\mathfrak{s}}^\sharp X^{\mathcal{M}}(\mathfrak{s})t(T_{\mathfrak{s}}^\sharp)_{\mathrm{rs}}}\big(\mathcal{H}\big(\mathrm{GL}_m(D)^\sharp\big)^{\mathfrak{s}^\sharp}\big)\big) \quad \text{and} \\ &\mathrm{Mod}_{f,(\mathfrak{t}_{\mathfrak{s}}^\sharp)_{\mathrm{rs}}}\big(\mathbb{H}(\mathfrak{t}_{\mathfrak{s}}^\sharp, W(R_{\mathfrak{s}})_t, q_{\mathfrak{s}}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\sharp}(t), \natural_{\mathfrak{s}^{\sharp\vee}}]\big). \end{aligned}$$

Every finite length representation in $\mathrm{Rep}(\mathrm{SL}_m(D))^{\mathfrak{s}^\sharp}$ decomposes canonically as a direct sum of generalized weight spaces for $\mathcal{O}(T_{\mathfrak{s}}^\sharp)^{X^{\mathcal{M}}(\mathfrak{s})W_{\mathfrak{s}}^\sharp}$, so by varying t in $(T_{\mathfrak{s}}^\sharp)_{\mathrm{un}}$ we can describe all such representations in terms of these equivalences of categories. In this sense

$$\mathbb{H}(\mathfrak{t}_{\mathfrak{s}}^\sharp, W(R_{\mathfrak{s}})_t, q_{\mathfrak{s}}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\sharp}(t), \natural_{\mathfrak{s}^{\sharp\vee}}] \tag{103}$$

is the graded Hecke algebra attached to $(\mathfrak{s}^\sharp, t)$. Suppose that t corresponds to $(\phi_b^\sharp, \rho^\sharp) \in \Phi_{\mathrm{cusp}}(\mathcal{L}^\sharp(F))$, where $\mathcal{M} = \mathcal{L}(F)$. Then we can compare (103) with (95). using the earlier comparison results (i), (ii) and (iii), we see that (103) is the specialization of (95) and $\vec{\mathbf{r}} = \log(q_{\mathfrak{s}})$.

We conclude that, for a Bernstein component $\mathfrak{s}^\sharp$ of $\mathrm{SL}_m(D)$, corresponding to a Bernstein component $\mathfrak{s}^{\sharp\vee}$ of enhanced L-parameters:

- The twisted graded Hecke algebras attached to $\mathfrak{s}^\sharp$ and to $\mathfrak{s}^{\sharp\vee}$ are isomorphic.
- The twisted affine Hecke algebras attached to $\mathfrak{s}^\sharp$ and to $\mathfrak{s}^{\sharp\vee}$ need not be isomorphic, but they are sufficiently close, so that their categories of finite length modules are equivalent.

5.3. **Pure inner twists of classical groups.**

Take $n \in \mathbb{N}$ and let $\mathcal{G}_n^*$ be a split connected classical group defined over F of rank n, that is, $\mathcal{G}_n^*$ is one the following groups

(i) Sp_{2n}, the symplectic group in $2n$ variables defined over F,
(ii) SO_{2n+1}, the split special orthogonal group in $2n+1$ variables defined over F,
(iii) SO_{2n}, the split special orthogonal group in $2n$ variables defined over F,

Let V^* be a finite dimensional F-vector space equipped with a non-degenerate symplectic or orthogonal form such that $\mathcal{G}_n^*(F)$ equals $\mathrm{Sp}(V^*)$ or $\mathrm{SO}(V^*)$. The pure inner twists $\mathcal{G}_n$ of $\mathcal{G}_n^*$ correspond bijectively to forms V of the space V^* with its bilinear form $\langle\,,\,\rangle$ [KMRT, §29D–E]. If $\mathcal{G}_n^*(F) = \mathrm{Sp}(V^*)$, then the pointed set $H_1(F, \mathcal{G}_n^*)$ has only one element and there are no nontrivial pure inner twists of $\mathcal{G}_n^*$. If $\mathcal{G}_n^*(F) = \mathrm{SO}(V^*)$, then elements of $H_1(F, \mathcal{G}_n^*)$ correspond bijectively to the isomorphism classes of orthogonal spaces V over F with $\dim(V) = \dim(V^*)$ and $\mathrm{disc}(V) = \mathrm{disc}(V^*)$. The corresponding pure inner twist of $\mathcal{G}_n^*(F)$ is the special orthogonal group $\mathrm{SO}(V)$.

Let $\mathcal{G}_n(F)$ be a pure inner twist of $\mathcal{G}_n^*(F)$ (we allow $\mathcal{G}_n(F) = \mathcal{G}_n^*(F)$). It is known (see for instance [ChGo]), that up to conjugacy every Levi subgroup of $\mathcal{G}_n(F)$ is of the form

$$\mathcal{L}(F) = \mathcal{G}_{n^-}(F) \times \prod\nolimits_j \mathrm{GL}_{m_j}(F), \tag{104}$$

where $\sum_j m_j + n^- = n$ and $\mathcal{G}_{n^-}(F)$ is an inner twist of the quasi-split connected classical group $\mathcal{G}^*_{n^-}$ defined over F, of rank n^-, which has the same type as $\mathcal{G}^*_n(F)$. There is a natural embedding $\mathrm{Std}_{^L\mathcal{G}}$ of $^L\mathcal{G}$ into $\mathrm{GL}_{N^\vee}(\mathbb{C}) \rtimes \mathbf{W}_F$, where $N^\vee = 2n+1$ if $\mathcal{G}^*_n = \mathrm{Sp}_{2n}$, and $N^\vee = 2n$ otherwise.

Let $(\phi, \rho) \in \Phi_{\mathrm{cusp}}(\mathcal{L}(F))$. The factorization (104) leads to

$$\mathrm{Std}_{^L\mathcal{G}} \circ \phi = \varphi \oplus \bigoplus\nolimits_j (\phi_j \oplus \phi_j^\vee). \tag{105}$$

Because we consider only pure inner twists in this section, instead of the group $\mathcal{S}_\phi$ defined in Definition 3.2, we will take the component group $\pi_0(Z_{\mathcal{L}^\vee}(\phi))$ and we use a variation on $\Phi_e(^L\mathcal{G})$ with that component group. The restriction of an enhancement ρ to the center of $\mathcal{L}^\vee$ still determines the relevance. For instance, if the restriction to $Z(\mathcal{L}^\vee)$ is trivial, then it corresponds to the split form, otherwise it corresponds to a non-split form. Hence, we can decompose $\rho = \varrho \otimes \bigotimes_j \rho_j$, where $(\varphi, \varrho) \in \Phi_{\mathrm{cusp}}(\mathcal{G}_{n^-}(F))$ and $(\phi_j, \rho_j) \in \Phi_{\mathrm{cusp}}(\mathrm{GL}_{m_j}(F))$ for each j.

Let I^{O}_ϕ (resp. I^{S}_ϕ) be the set of (classes of) self-dual irreducible representations of $\mathbf{W}_F$ which occur in $\mathrm{Std}_{^L\mathcal{G}} \circ \phi$ and which factor through a group of the type of $\mathcal{G}^\vee$ (resp. of opposite type of $\mathcal{G}^\vee$). Let I^{GL}_ϕ be a set of (classes of) non self-dual irreducible representations of $\mathbf{W}_F$ which occur in $\mathrm{Std}_{^L\mathcal{G}} \circ \phi$, such that if $\phi \in I^{\mathrm{GL}}_\phi$ then $\phi^\vee \notin I^{\mathrm{GL}}_\phi$, and maximal for this property.

On the one hand (ϕ_j, ρ_j) satisfy the conditions stated in Paragraph 5.1, i.e. ϕ_j is an irreducible representation of $\mathbf{W}_F$ and ρ_j is the trivial representation of $\pi_0(Z_{\mathrm{GL}_{m_j}(\mathbb{C})}(\phi_j))$. On the other hand, after [Mou, Proposition 3.7], we have :

$$\mathrm{Std}_{^L\mathcal{G}_{n^-}} \circ \varphi = \bigoplus_{\tau \in I^{\mathrm{O}}_\varphi} \bigoplus_{i=1}^{a_\tau} (\tau \otimes S_{2i-1}) \oplus \bigoplus_{\tau \in I^{\mathrm{S}}_\varphi} \bigoplus_{i=1}^{a_\tau} (\tau \otimes S_{2i}), \tag{106}$$

where the a_τ are non-negative integers. As it was introduced by Mœglin, let $\mathrm{Jord}(\varphi)$ be the set of pairs (τ, a) with $\tau \in \mathrm{Irr}(\mathbf{W}_F)$, $a \in \mathbb{N}^*$ such that $\tau \boxtimes S_a$ is an irreducible subrepresentation of $\mathrm{Std}_{^L\mathcal{G}_{n^-}} \circ \varphi$.

The group $\mathcal{S}_\phi$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^p$ for some integer p and then generated by elements of order two $z_{\tau,a} z_{\tau,a'}$ with $(\tau, a), (\tau, a') \in \mathrm{Jord}(\phi)$ without hypothesis on the parity of a and by $z_{\tau,a}$ when a is even. The character ρ satisfies $\rho(z_{\tau,2i-1} z_{\tau,2i+1}) = -1$ for all $\tau \in I^{\mathrm{O}}_\varphi$ and $i \in [\![1, \frac{a_\tau - 1}{2}]\!]$ and $\rho(z_{\tau,2i}) = (-1)^i$ for all $\tau \in I^{\mathrm{S}}_\varphi$ and $i \in [\![1, \frac{a_\tau}{2}]\!]$.

We begin by computing the group $W^\circ_{\mathfrak{s}^\vee}$, so let us consider the restriction of ϕ to $\mathbf{I}_F$. If τ is an irreducible representation of $\mathbf{W}_F$ and of dimension m such that $\tau|_{\mathbf{I}_F} \simeq \tau^\vee|_{\mathbf{I}_F}$, then $\tau \simeq \tau^\vee z$ with $z \in X_{\mathrm{nr}}(^L\mathrm{GL}_m(F))$. Replacing τ by $\tau z^{1/2}$ (where $z^{1/2}$ is any square root of z), we can assume that $\tau \simeq \tau^\vee$. In the following, for all j we assume that, if $\phi_j^\vee$ is inertially equivalent to ϕ_j, then $\phi_j^\vee \simeq \phi_j$. Note that a self-dual irreducible representation of $\mathbf{W}_F$ is necessarily of symplectic-type or of orthogonal-type.

For an irreducible representation τ of $\mathbf{W}_F$, we will denote by e_τ the number of times where τ appears in a GL factor of $\mathcal{L}^\vee$, by ℓ_τ the multiplicity of τ in $\varphi|_{\mathbf{W}_F}$ and by $I^{\mathrm{O}}_{\phi,\mathrm{GL}}$ (resp. $I^{\mathrm{S}}_{\phi,\mathrm{GL}}$) the set of such τ which are of orthogonal-type (resp.

symplectic-type). Recall that $\mathcal{L}^\vee = \mathcal{G}_{n^-}^\vee(\mathbb{C}) \times \prod_j \mathrm{GL}_{m_j}(\mathbb{C})$. Now we can write :

$$\begin{aligned}\phi|_{\mathbf{I}_F} &= \varphi|_{\mathbf{I}_F} \oplus \bigoplus_j (\phi_j|_{\mathbf{I}_F} \oplus \phi_j^\vee|_{\mathbf{I}_F}) \\ &= \left(\bigoplus_{\tau\in I_\varphi^{\mathrm{O}}\sqcup I_\varphi^{\mathrm{S}}} \ell_\tau\tau|_{\mathbf{I}_F}\right) \oplus \bigoplus_{\tau\in\tau\in I_{\phi,\mathrm{GL}}^{\mathrm{O}}\sqcup I_{\phi,\mathrm{GL}}^{\mathrm{S}}} 2e_\tau\tau|_{\mathbf{I}_F} \oplus \bigoplus_{\tau\in I_\phi^{\mathrm{GL}}} e_\tau(\tau|_{\mathbf{I}_F}\oplus\tau^\vee|_{\mathbf{I}_F}) \\ &= \bigoplus_{\tau\in I_\phi^{\mathrm{O}}\sqcup I_\phi^{\mathrm{S}}} (2e_\tau+\ell_\tau)\tau|_{\mathbf{I}_F} \oplus \bigoplus_{\tau\in I_\phi^{\mathrm{GL}}} e_\tau(\tau|_{\mathbf{I}_F}\oplus\tau^\vee|_{\mathbf{I}_F})\end{aligned}$$

We have assumed that for $\tau \in I_\phi^{\mathrm{GL}}$, $\tau|_{\mathbf{I}_F} \not\simeq \tau^\vee|_{\mathbf{I}_F}$ and we know that an irreducible representation τ of $\mathbf{W}_F$, decomposes to $\mathbf{I}_F$ as $\tau|_{\mathbf{I}_F} = \theta \oplus \theta^{\mathrm{Frob}_F} \oplus \ldots \oplus \theta^{\mathrm{Frob}_F^{s_\tau-1}}$, with θ an irreducible representation of $\mathbf{I}_F$ and for all $w \in \mathbf{I}_F$, $\theta^{\mathrm{Frob}_F^k}(w) = \theta(\mathrm{Frob}_F^{-k} w \mathrm{Frob}_F^k)$. If we assume $\tau|_{\mathbf{I}_F} \simeq \tau^\vee|_{\mathbf{I}_F}$, then $\theta^\vee \simeq \theta^{\mathrm{Frob}_F^i}$ for some integer i between 0 and $s_\tau - 1$. Then we have $\theta \simeq \theta^{\mathrm{Frob}_F^i\,\vee} \simeq \theta^{\mathrm{Frob}_F^{2i}}$. This implies that $i = 0$ or s_τ is even and $i = s_\tau/2$. In the first case, $\theta^\vee \simeq \theta$ and in the second case $\theta^\vee \simeq \theta^{\mathrm{Frob}_F^{s_\tau/2}}$. In the second case, $\tau \in I_\phi^{\mathrm{S}}$. Indeed, let us denote by $n_\tau = 2e_\tau + \ell_\tau$ and by N_τ the multiplicity space of $\tau^{\oplus n_\tau}$. We already know that $N_\tau^\vee \simeq N_\tau$ and an isomorphism is given by $f : N_\tau \to N_\tau^\vee$, $A \mapsto A^\vee$. The intertwining operator A permutes the subspaces of τ as

$$\theta^{\oplus n_\tau} \to \theta^{\mathrm{Frob}_F\,\oplus n_\tau} \to \ldots \to \theta^{\mathrm{Frob}_F^{s_\tau-1}\,\oplus n_\tau}.$$

We can then see that $A^\vee$ acts as

$$\theta^{\vee\oplus n_\tau} \to \theta^{\mathrm{Frob}_F\,\vee\oplus n_\tau} \to \ldots \to \theta^{\mathrm{Frob}_F^{s_\tau-1}\,\vee\oplus n_\tau},$$

that is,

$$\theta^{\mathrm{Frob}_F^{s_\tau/2}\,\oplus n_\tau} \to \theta^{\mathrm{Frob}_F^{s_\tau/2+1}\,\oplus n_\tau} \to \ldots \to \theta^{\mathrm{Frob}_F^{s_\tau/2-1}\,\oplus n_\tau}.$$

Since the order of Frob_F is s_τ, $\mathrm{Frob}_F^{s_\tau/2}$ acts by -1 and $f^\vee = -f$. So $\tau \in I_\phi^{\mathrm{S}}$. We denote by $I_\phi^{\mathrm{S},1}$ the set of all $\tau \in I_\phi^{\mathrm{S}}$ which satisfy the second property, and by $I_\phi^{\mathrm{S},0}$ the complementary set in I_ϕ^{S}. So we find that :

$$\begin{aligned}&J^\circ = Z_{\mathcal{G}^\vee}(\phi|_{\mathbf{I}_F})^\circ \simeq \\ &\prod_{\tau\in I_\phi^{\mathrm{O}}} \mathrm{SO}_{2e_\tau+\ell_\tau}(\mathbb{C})^{s_\tau} \times \prod_{\tau\in I_\phi^{\mathrm{S},0}} \mathrm{Sp}_{2e_\tau+\ell_\tau}(\mathbb{C})^{s_\tau} \times \prod_{\tau\in I_\phi^{\mathrm{S},1}} \mathrm{GL}_{2e_\tau+\ell_\tau}(\mathbb{C})^{s_\tau} \times \prod_{\tau\in I_\phi^{\mathrm{GL}}} \mathrm{GL}_{e_\tau}(\mathbb{C})^{s_\tau}.\end{aligned}$$

For all $\tau \in I_\phi^{\mathrm{O}}$, we have an embedding of $(\mathbb{C}^\times)^{e_\tau}$ into $(\mathbb{C}^\times)^{e_\tau} \times \mathrm{SO}_{\ell_\tau}(\mathbb{C})$ and the last one is embedded diagonally as Levi subgroup in $\mathrm{SO}_{2e_\tau+\ell_\tau}(\mathbb{C})^{s_\tau}$. We have the same kind of embedding for $\tau \in I_\phi^{\mathrm{S},0}$ or $\tau \in I_\phi^{\mathrm{GL}}$. For $\tau \in I_\phi^{\mathrm{S},1}$, the embedding of $(\mathbb{C}^\times)^{e_\tau}$ in $\mathrm{GL}_{2e_\tau+\ell_\tau}(\mathbb{C})$ is given by $(z_1, \ldots, z_{e_\tau}) \mapsto \mathrm{diag}(z_1, \ldots, z_{e_\tau}, 1, \ldots, 1, z_{e_\tau}^{-1}, \ldots, z_1^{-1})$, with ℓ_τ times 1. Moreover $R(J^\circ, T)$ is a union of irreducible components corresponding to each τ with :

	condition	R_τ	$R_{\tau,\mathrm{red}}$
$\tau \in I_\phi^{\mathrm{S}}$	$e = 0$	$\varnothing$	$\varnothing$
	$e \neq 0, \ell = 0$	C_e	C_e
	$e \neq 0, \ell \neq 0$	BC_e	B_e
$\tau \in I_\phi^{\mathrm{O}}$	$e = 0$	$\varnothing$	$\varnothing$
	$e \neq 0, \ell = 0$	D_e	D_e
	$e \neq 0, \ell \neq 0$	B_e	B_e
$\tau \in I_\phi^{\mathrm{GL}}$	$e \leqslant 1$	$\varnothing$	$\varnothing$
	$e \geqslant 2$	A_{e-1}	A_{e-1}

This tells us that that we can make a choice of basepoint for ϕ as follows :

- if $m_i = m_j$ and ϕ_i differs from ϕ_j by an unramified twist, then $\phi_i = \phi_j$;
- if $\phi_i^\vee$ is an unramified twist of ϕ_i, then we can assume that $\phi_i^\vee \simeq \phi_i$;
- if $\phi_i^\vee \simeq \phi_j$, then $i = j$.

For this choice of ϕ, for all $\tau \in I_\phi^{\mathrm{O}} \sqcup I_\phi^{\mathrm{S}} \sqcup I_\phi^{\mathrm{GL}}$, recall that e_τ is the number of τ which occurs in the GL factors and ℓ_τ is the multiplicity of τ in φ. Hence we have :

$$\begin{aligned}
\phi &= \bigoplus_{\tau \in I_\phi^{\mathrm{O}} \sqcup I_\phi^{\mathrm{S}}} 2e_\tau \tau \oplus \bigoplus_{\tau \in I_\phi^{\mathrm{GL}}} e_\tau(\tau \oplus \tau^\vee) \oplus \varphi, \\
\phi|_{\mathbf{W}_F} &= \bigoplus_{\tau \in I_\phi^{\mathrm{O}} \sqcup I_\phi^{\mathrm{S}}} (2e_\tau + \ell_\tau)\tau \oplus \bigoplus_{\tau \in I_\phi^{\mathrm{GL}}} e_\tau(\tau \oplus \tau^\vee), \\
G_\phi &\cong \prod_{\tau \in I_\phi^{\mathrm{S}}} \mathrm{Sp}_{2e_\tau + \ell_\tau}(\mathbb{C}) \times S\left(\prod_{\tau \in I_\phi^{\mathrm{O}}} \mathrm{O}_{2e_\tau + \ell_\tau}(\mathbb{C})\right) \times \prod_{\tau \in I_\phi^{\mathrm{GL}}} \mathrm{GL}_{e_\tau}(\mathbb{C}), \\
M &\cong \prod_{\tau \in I_\phi^{\mathrm{S}}} (\mathbb{C}^\times)^{e_\tau} \times \mathrm{Sp}_{\ell_\tau}(\mathbb{C}) \times S\left(\prod_{\tau \in I_\phi^{\mathrm{O}}} (\mathbb{C}^\times)^{e_\tau} \times \mathrm{O}_{\ell_\tau}(\mathbb{C})\right) \times \prod_{\tau \in I_\phi^{\mathrm{GL}}} (\mathbb{C}^\times)^{e_\tau}.
\end{aligned}$$

The above expression for G_ϕ° naturally factors as $\prod_{\tau \in I_\phi^{\mathrm{S}} \sqcup I_\phi^{\mathrm{O}} \sqcup I_\phi^{\mathrm{GL}}} G_\tau^\circ$. With that we can write

$$T \cong \prod_{\tau \in I_\phi^{\mathrm{S}} \sqcup I_\phi^{\mathrm{O}} \sqcup I_\phi^{\mathrm{GL}}} (\mathbb{C}^\times)^{e_\tau}, \quad R(G_\phi^\circ, T) \cong \prod_\tau R(G_\tau^\circ T, T).$$

After [Mou, §4.1], in the following table we describe the root systems and the Weyl groups. We abbreviate $R_\tau = R(G_\tau^\circ T, T)$ and $R_{\tau,\mathrm{red}} = R(G_\tau^\circ T, T)_{\mathrm{red}}$.

G_τ	M_τ	condition	R_τ	$R_{\tau,\mathrm{red}}$	$W_{M_\tau^\circ}^{G_\tau^\circ}$	$W_{M_\tau}^{G_\tau}$
$\mathrm{Sp}_{2e+\ell}(\mathbb{C})$	$(\mathbb{C}^\times)^e \times \mathrm{Sp}_\ell(\mathbb{C})$	$e=0$	$\varnothing$	$\varnothing$	$\{1\}$	$\{1\}$
		$e \neq 0, \ell = 0$	C_e	C_e	$S_e \rtimes (\mathbb{Z}/2\mathbb{Z})^e$	$S_e \rtimes (\mathbb{Z}/2\mathbb{Z})^e$
		$e \neq 0, \ell \neq 0$	BC_e	B_e	$S_e \rtimes (\mathbb{Z}/2\mathbb{Z})^e$	$S_e \rtimes (\mathbb{Z}/2\mathbb{Z})^e$
$\mathrm{O}_m(\mathbb{C})$	$(\mathbb{C}^\times)^e \times \mathrm{O}_\ell(\mathbb{C})$	$e=0$	$\varnothing$	$\varnothing$	$\{1\}$	$\{1\}$
		$e \neq 0, \ell = 0$	D_e	D_e	$S_e \rtimes (\mathbb{Z}/2\mathbb{Z})^{e-1}$	$S_e \rtimes (\mathbb{Z}/2\mathbb{Z})^e$
		$e \neq 0, \ell \neq 0$	B_e	B_e	$S_e \rtimes (\mathbb{Z}/2\mathbb{Z})^e$	$S_e \rtimes (\mathbb{Z}/2\mathbb{Z})^e$
$\mathrm{GL}_e(\mathbb{C})$	$(\mathbb{C}^\times)^e$	$e \leqslant 1$	$\varnothing$	$\varnothing$	$\{1\}$	$\{1\}$
		$e \geqslant 2$	A_{e-1}	A_{e-1}	S_e	S_e

For all $\tau \in I_\phi^{\mathrm{O}}$, such that $\ell_\tau = 0$, let $r_\tau \in W_{M_\tau}^{G_\tau} \setminus W_{M_\tau^\circ}^{G_\tau^\circ}$. The finite group $\mathfrak{R}_{\mathfrak{s}^\vee}$ is the subgroup $\langle r_\tau \mid \tau \in I_\phi^{\mathrm{O}}, \ell_\tau = 0\rangle$. More precisely, let

$$\begin{aligned} C &= \{\tau \in I_\phi^{\mathrm{O}} \mid \ell_\tau = 0\},\\ C_{even} &= \{\tau \in C \mid \dim\tau \equiv 0 \mod 2\},\\ C_{odd} &= \{\tau \in C \mid \dim\tau \equiv 1 \mod 2\}. \end{aligned}$$

It was shown in [Mou, §4.1] that :

- if $\mathcal{G} = \mathrm{Sp}_N$ or $\mathcal{G} = \mathrm{SO}_N$ with N odd, then
$$\mathfrak{R}_{\mathfrak{s}^\vee} \cong \prod_{\tau\in C} \langle r_\tau\rangle;$$
- if $\mathcal{G} = \mathrm{SO}_N$ and $\mathcal{L} = \mathrm{GL}_{d_1}^{\ell_1} \times \ldots \times \mathrm{GL}_{d_r}^{\ell_r} \times \mathrm{SO}_{N'}$ with N even and $N' \geqslant 4$, then
$$\mathfrak{R}_{\mathfrak{s}^\vee} \cong \prod_{\tau\in C} \langle r_\tau\rangle;$$
- if $\mathcal{G} = \mathrm{SO}_N$ and $\mathcal{L} = \mathrm{GL}_{d_1}^{\ell_1} \times \ldots \times \mathrm{GL}_{d_r}^{\ell_r}$ with N even, then
$$\mathfrak{R}_{\mathfrak{s}^\vee} \cong \prod_{\tau\in C_{even}} \langle r_\tau\rangle \times \langle r_\tau r_{\tau'} \mid \tau,\tau' \in C_{odd}\rangle.$$

So far we have described $W_{\mathfrak{s}^\vee}^\circ$ and $\mathfrak{R}_{\mathfrak{s}^\vee}$; let us describe the parameter function. For that, from the shape of M_τ°, we can describe the unipotent element v_τ in the following table :

M_τ°	v_τ	ℓ
$(\mathbb{C}^\times)^e \times \mathrm{Sp}_\ell(\mathbb{C})$	$(1^e) \times (2,4,\ldots,2d-2,2d)$	$\ell = d(d+1)$
$(\mathbb{C}^\times)^e \times \mathrm{O}_\ell(\mathbb{C})$	$(1^e) \times (1,3,\ldots,2d-3,2d-1)$	$\ell = d^2$
$(\mathbb{C}^\times)^e$	(1^e)	

To be complete, let us describe the cuspidal representation of $A_{M_\tau^\circ}(v_\tau)$. First, we have :

$$A_{M_\tau^\circ}(v_\tau) \simeq \begin{cases} (\mathbb{Z}/2\mathbb{Z})^d = \langle z_{\tau,2a}, a \in [\![1,d]\!]\rangle & \text{if } \tau \in I_\phi^{\mathrm{S}} \\ (\mathbb{Z}/2\mathbb{Z})^{d-1} = \langle z_{\tau,2a-1} z_{\tau,2a+1}, a \in [\![1,d-1]\!]\rangle & \text{if } \tau \in I_\phi^{\mathrm{O}} \end{cases}.$$

Moreover, the cuspidal irreducible representation ϵ_τ of $A_{M_\tau^\circ}(v_\tau)$ satisfies $\epsilon_\tau(z_{\tau,2a}) = (-1)^a$ if $\tau \in I_\phi^{\mathrm{S}}$ and $\epsilon_\tau(z_{\tau,2a-1}z_{\tau,2a+1}) = -1$ if $\tau \in I_\phi^{\mathrm{O}}$. For all $\tau \in I_\phi^{\mathrm{O}} \sqcup I_\phi^{\mathrm{S}}$, denote by a_τ the biggest part of the partition of v_τ and by a'_τ the biggest part of the partition of $v_{\xi\tau}$, where ξ is an unramified character such that $(\xi\tau)^\vee \simeq \xi\tau$ and $\xi\tau \not\simeq \tau$. In case $v_{\xi\tau} = 1$, then we will assume that $a'_\tau = 0$ if $\tau \in I_\phi^{\mathrm{S}}$ and $a'_\tau = -1$ if $\tau \in I_\phi^{\mathrm{O}}$ (this follows from Lemma 3.11).

When $R_{\tau,\mathrm{red}}$ is of type A, C or D, then after [Lus2, 2.13] for all simple roots $\alpha \in R_{\tau,\mathrm{red}}$, $c(\alpha) = 2$, so $\lambda(\alpha) = 1$. When $R_{\tau,\mathrm{red}}$ is of type B, then for all simple roots which are not short $\lambda(\alpha) = 1$. For the simple short root $\alpha_\tau \in R_{\tau,\mathrm{red}}$, we have, $c(\alpha_\tau) = a_\tau + 1$ and $c^*(\alpha_\tau) = a'_\tau + 1$, hence

$$\lambda(\alpha_\tau) = \frac{a_\tau + a'_\tau}{2} + 1 \text{ and } \lambda^*(\alpha_\tau) = \frac{|a_\tau - a'_\tau|}{2}.$$

We conclude that

$$\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) = \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}] \cong \bigotimes\nolimits_\tau \mathcal{H}(G_\tau^\circ, M_\tau^\circ, v_\tau, \epsilon_\tau, \mathbf{z}_\tau) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}]. \tag{107}$$

Let $t(\tau)$ be the order of the group of unramified characters χ such that $\tau \simeq \tau\chi$. Via the specialization of $\mathbf{z}_\tau$ at $q_F^{t(\tau)/2}$, (107) becomes precisely the extended affine Hecke algebra given in [Hei2].

Moreover, it was shown in [Hei2] that there is an equivalence of categories between $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$ and the right modules over $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) / (\{\mathbf{z}_\tau - q_F^{t(\tau)/2}\}_\tau)$. Together with the LLC for $\mathcal{G}(F)$ we get bijections

$$\mathrm{Irr}\Big(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) / \big(\{\mathbf{z}_\tau - q_F^{t(\tau)/2}\}_\tau\big)\Big) \longleftrightarrow \mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}} \longleftrightarrow \Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}.$$

It does not seem unlikely that this works out to the same bijection as in Theorem 3.15.a. But at present that is hard to check, because the LLC is not really explicit.

Example 5.2. We consider an example that illustrates many of the above aspects. Let $\tau : \mathbf{W}_F \to \mathrm{GL}_4(\mathbb{C})$ be an irreducible represetation of $\mathbf{W}_F$, self-dual of symplectic-type and let $\varphi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{37}(\mathbb{C})$ defined by

$$\varphi = 1 \boxtimes (S_5 \oplus S_3 \oplus S_1) \oplus \xi \boxtimes (S_3 \oplus S_1) \oplus \tau \boxtimes (S_4 \oplus S_2),$$

with $\xi : \mathbf{W}_F \to \mathbb{C}^\times$ an unramified quadratic character. We have $Z_{\mathrm{SO}_{37}(\mathbb{C})}(\varphi|_{\mathbf{W}_F})^\circ \simeq \mathrm{SO}_9(\mathbb{C}) \times \mathrm{SO}_4(\mathbb{C}) \times \mathrm{Sp}_6(\mathbb{C})$, and φ defines a L-packet $\Pi_\varphi(\mathrm{Sp}_{36}(F))$ with 2^6 elements and 2 of them are supercuspidal.

Let $\sigma \in \Pi_\varphi(\mathrm{Sp}_{36}(F))$ supercuspidal corresponding to an enhanced Langlands parameter (φ, ε) with ε cuspidal. Consider $\mathcal{G}(F) = \mathrm{Sp}_{58}(F)$, the Levi subgroup $\mathcal{L}(F) = \mathrm{GL}_4(F)^2 \times \mathrm{GL}_1(F)^3 \times \mathrm{Sp}_{36}(F)$ and $\tau^{\otimes 2} \boxtimes 1^{\otimes 3} \boxtimes \sigma$ be the irreducible supercuspidal representation of $\mathcal{L}(F)$. The inertial pair $\mathfrak{s} = [\mathcal{L}(F), \tau^{\otimes 2} \boxtimes 1^{\otimes 3} \boxtimes \sigma]$ of $\mathcal{G}(F)$ admits $\mathfrak{s}^\vee = [\mathcal{L}^\vee, \phi, \varepsilon]$ as dual inertial triple with $\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathcal{L}^\vee$, with $\mathrm{Std}_{\mathcal{L}^\vee} \circ \phi = (\tau \oplus \tau^\vee)^{\oplus 2} \oplus (1 \oplus 1^\vee)^{\oplus 3} \oplus \varphi$. We assume that $\tau|_{\mathbf{I}_F} = \theta \oplus \theta^{\mathrm{Frob}_F}$ with $\theta^\vee \simeq \theta$.

We first compute $W_{\mathfrak{s}^\vee}^\circ$:

$$\phi|_{\mathbf{I}_F} = \tau|_{\mathbf{I}_F}^{\oplus 4} \oplus 1|_{\mathbf{I}_F}^{\oplus 6} \oplus 1|_{\mathbf{I}_F}^{\oplus 9} \oplus \xi|_{\mathbf{I}_F}^{\oplus 4} \oplus \tau|_{\mathbf{I}_F}^{\oplus 6} = \theta^{\oplus 10} \oplus \theta^{\mathrm{Frob}_F \oplus 10} \oplus 1^{\oplus 19},$$
$$J^\circ = Z_{\mathcal{G}^\vee}(\phi|_{\mathbf{I}_F})^\circ \simeq \mathrm{Sp}_{10}(\mathbb{C})^2 \times \mathrm{SO}_{19}(\mathbb{C}).$$

The torus T is decomposed as $T = (\mathbb{C}^\times)^2 \times (\mathbb{C}^\times)^3$. The first part $(\mathbb{C}^\times)^2$ is embedded in an obvious way in $(\mathbb{C}^\times)^2 \times \mathrm{Sp}_6(\mathbb{C})$ and then in $\mathrm{Sp}_{10}(\mathbb{C})^2$ diagonally as Levi subgroup. The second part $(\mathbb{C}^\times)^3$ is embedded in $(\mathbb{C}^\times)^3 \times \mathrm{SO}_{13}(\mathbb{C})$ and then in $\mathrm{SO}_{19}(\mathbb{C})$ as Levi subgroup as well. The root system $R(J^\circ, T)$ (resp. $R(J^\circ, T)_{\mathrm{red}}$) is $BC_2 \times B_3$ (resp. $B_2 \times B_3$), so $W_{\mathfrak{s}^\vee}^\circ = W_{B_2} \times W_{B_3}$.

From the above discussion, we can see that ϕ is already a basepoint. If we denote by ϕ' the parameter defined by $\phi' = (\tau \oplus \tau^\vee)^{\oplus 2} \oplus (\xi \oplus \xi^\vee)^{\oplus 3} \oplus \varphi$, then ϕ' is an other basepoint. Indeed, we have :

$$\phi|_{\mathbf{W}_F} = \tau^{\oplus 10} \oplus 1^{\oplus 15} \oplus \xi^{\oplus 4}$$
$$G_\phi^\circ = Z_{\mathcal{G}^\vee}(\phi|_{\mathbf{W}_F})^\circ \simeq \mathrm{Sp}_{10}(\mathbb{C}) \times \mathrm{SO}_{15}(\mathbb{C}) \times \mathrm{SO}_4(\mathbb{C})$$
$$L_\phi^\circ = Z_{\mathcal{L}^\vee}(\phi|_{\mathbf{W}_F})^\circ \simeq ((\mathbb{C}^\times)^2 \times \mathrm{Sp}_6(\mathbb{C})) \times ((\mathbb{C}^\times)^3 \times \mathrm{SO}_9(\mathbb{C})) \times \mathrm{SO}_4(\mathbb{C})$$
$$\phi'|_{\mathbf{W}_F} = \tau^{\oplus 10} \oplus 1^{\oplus 9} \oplus \xi^{\oplus 10}$$
$$G_{\phi'}^\circ = Z_{\mathcal{G}^\vee}(\phi'|_{\mathbf{W}_F})^\circ \simeq \mathrm{Sp}_{10}(\mathbb{C}) \times \mathrm{SO}_9(\mathbb{C}) \times \mathrm{SO}_{10}(\mathbb{C})$$
$$L_{\phi'}^\circ = Z_{\mathcal{L}^\vee}(\phi'|_{\mathbf{W}_F})^\circ \simeq ((\mathbb{C}^\times)^2 \times \mathrm{Sp}_6(\mathbb{C})) \times \mathrm{SO}_9(\mathbb{C}) \times ((\mathbb{C}^\times)^3 \times \mathrm{SO}_4(\mathbb{C}))$$

Here $R_{\mathfrak{s}^\vee}$ is trivial, so $W_{\mathfrak{s}^\vee} = W_{\mathfrak{s}}^\circ$. Denote by α_1, α_2 (resp. $\beta_1, \beta_2, \beta_3$) the simple roots of B_2 (resp. B_3) with α_2 (resp. β_3) the short root. The parameters are given by $\lambda(\alpha_1) = \lambda(\beta_1) = \lambda(\beta_2) = 1$, $\lambda(\alpha_2) = \frac{4}{2} + 1 = 3$, $\lambda(\beta_3) = \frac{5+3}{2} + 1 = 5$ and $\lambda^*(\alpha_2) = \frac{4}{2} = 2$, $\lambda^*(\beta_3) = \frac{5-3}{2} = 1$. The quadratic relations in the Hecke algebra are

$$(T_{s_{\alpha_1}} - q^{t(\tau)})(T_{s_{\alpha_1}} + 1) = 0,\ (T_{s_{\beta_1}} - q)(T_{s_{\beta_1}} + 1) = 0,\ (T_{s_{\beta_2}} - q)(T_{s_{\beta_2}} + 1) = 0,$$
$$(T_{s_{\alpha_2}} - q^{3t(\tau)})(T_{s_{\alpha_2}} + 1) = 0,\ (T_{s_{\beta_3}} - q^5)(T_{s_{\beta_3}} + 1) = 0.$$

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