ARTHUR PACKETS AND ADAMS-BARBASCH-VOGAN PACKETS FOR p-ADIC GROUPS, PART 2

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Abstract. This paper shows – by examples – how to calculate the transfer coefficients that appear in Arthur's main local result in the endoscopic classification of representations, using purely geometric tools. Specifically, we use vanishing cycles of perverse sheaves to calculate examples of Adams-Barbasch-Vogan packets for p-adic groups and of endoscopic transfer and twisted endoscopic transfer of Adams-Barbasch-Vogan packets. By comparing these to Arthur packets we gather evidence for the conjecture that Arthur packets are Adams-Barbasch-Vogan packets. We also verify the Kazhdan-Lusztig conjecture for the admissible representations of p-adic groups that appear in our examples. The techniques we use here build on results from [7], but this paper also provides a bridge to some of the ideas used in [8] to prove that Arthur packets are Adams-Barbasch-Vogan packets for unipotent representations of general linear groups and for odd special orthogonal groups and their pure inner forms over p-adic fields.

Contents

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Introduction

Our objective in this paper is to show how to use vanishing cycles of perverse sheaves to calculate the local transfer coefficients $\langle s_\psi\, s, \pi\rangle_\psi$ that appear in Arthur's endoscopic classification [3, Theorem 1.5.1]. We do this by independently calculating both sides of [7, Conjecture 2] in examples:

$$\langle s_\psi\, s, \pi\rangle_\psi = (-1)^{\dim C_\psi - \dim C_\pi}\, \mathrm{trace}_s\, \mathsf{NEv}_\psi\, \mathcal{P}(\pi), \tag{1}$$

for every $s \in Z_{\widehat{G}}(\psi)$. By making these calculations, we wish to demonstrate that the functor NEv, introduced in [7], provides a practical tool for calculating Arthur packets, the associated stable distributions and their transfer under endoscopy. Our examples provide evidence for the conjecture that Arthur packets are Adams-Barbasch-Vogan packets as well as the Kazhdan-Lusztig conjecture for p-adic groups. This paper builds on results from [7] and as such, should be read with that paper in hand.

In this paper we consider admissible representations of the p-adic groups SL(2), PGL(4), SO(3), SO(5) and SO(7). There are a variety of reasons why we have chosen to present this specific set of examples. The groups SO(3), SO(5), and SO(7) are the first few groups in the family SO($2n+1$), and this is the family we study in [8] for unipotent representations. The group SO(7) is the first in this family to exhibit some of the more general phenomena that meaningfully illuminate the conjectures from [7]. Moreover, since SO(3) $\times$ SO(3) is an elliptic endoscopic group for SO(5) and SO(5) $\times$ SO(3) is an elliptic endoscopic group for SO(7), we are also able to use these examples to show how to use geometric tools to compute Langlands-Shelstad transfer of invariant distributions for endoscopic groups. Not only was this ultimately a useful feature for doing the geometric calculations, but presenting these examples side by side allows one to see certain relationships that hold more generally for endoscopic groups. We also include two examples – for SL(2) and

PGL(4) – that show how the problem of calculating Arthur packets and Arthur's transfer coefficients is reduced to unipotent representations.

Each example follows essentially the same four-part plan, explained in some detail in Section 0 and outlined here.

(§0.1) After fixing a connected reductive group G over a p-adic field F from the list above and an infinitesimal parameter $\lambda : W_F \to {}^L G$, we enumerate all admissible representations π of all pure rational forms of G with infinitesimal parameter λ. We partition these admissible representations into L-packets and show how Aubert duality operates on the representations. Then, for each L-packet of Arthur type, we find the Arthur packet that contains it. We calculate a twisting character which measures the difference between Arthur's parametrization of representations in an Arthur packet with Mœglin's parametrization. We find the coefficients in the invariant distributions

$$\Theta^G_{\psi,s} = \sum_{\pi \in \Pi_\psi(G(F))} \langle s\, s_\psi, \pi \rangle_\psi \operatorname{trace} \pi \tag{2}$$

that arise from stable distributions attached to Arthur packets for endoscopic groups for $G(F)$ in [3, Theorem 1.5.1]. We also calculate the virtual representations $\eta_{\psi,s}$ defined in [7, Section 1.11] using Arthur's work. See Section 0.1 for more detail on this part of the examples.

(§0.2) In the second part of each example, called *Vanishing cycles of perverse sheaves*, we set up all the tools needed to calculate $\langle s s_\psi, \pi \rangle_\psi$, and its generalisation to pure rational forms of G, geometrically. We find the stratified variety V_λ attached to λ and study the category $\mathsf{Per}_{Z_{\widehat{G}}(\lambda)}(V_\lambda)$ of equivariant perverse sheaves on V_λ. We show how this category decomposes into summand categories, called the cuspidal support decomposition of $\mathsf{Per}_{Z_{\widehat{G}}(\lambda)}(V_\lambda)$. Then we calculate the functor

$$\mathsf{Ev}_\psi : \mathsf{Per}_{Z_{\widehat{G}}(\lambda)}(V_\lambda) \to \mathsf{Rep}(A_\psi) \tag{3}$$

on simple objects, using properties of vanishing cycles; NEv is defined in [7, Section 5] and recalled in Section 0.2.6. The results of these calculations – one for each example – are presented in Sections 1.2.3, 2.2.5, 3.2.5, 4.2.5, 5.2.5 and 6.2.6. Section 0.2 includes an overview of how we made these calculations. We also show how the Fourier transform interacts with the functor NEv.

(§0.3) In the third part we connect the two sides of this story, as treated above. To begin, we find Vogan's bijection between: admissible representations of split p-adic groups and their pure rational forms with fixed infinitesimal parameter $\lambda : W_F \to {}^L G$, as recalled in Section 0.1; and simple equivariant perverse sheaves on V_λ, as recalled in Section 0.2. With this bijection in hand, and the calculation of Ev from Section 0.2, we easily find the Adams-Barbash-Vogan packets $\Pi^{\mathrm{ABV}}_{\mathrm{pure},\phi}$ and associated virtual representations $\eta^{\mathsf{NEv}}_{\phi,s}$. By referring back to Section 0.1, we easily see

$$\eta_{\psi,s} = \eta^{\mathsf{NEv}}_{\phi_\psi,s} \tag{4}$$

for all Arthur parameters ψ with infinitesimal parameter λ, thus confirming [7, Conjecture 2] in the examples. This implies (1) and also implies

$$\Pi_{\mathrm{pure},\psi} = \Pi^{\mathrm{ABV}}_{\mathrm{pure},\phi_\psi} \tag{5}$$

for every Arthur parameter with infinitesimal parameter λ. We also verify the Kazhdan-Lusztig conjecture in each example, which allows us to verify [7, Conjecture 3] in our examples. We show how the twisting characters χ_ψ from Section 0.1.5 relate to the twisting local system $\mathcal{T}_\psi$ introduced in [7] and recalled in Section 0.2.8. While (5) shows that every Arthur packet is an ABV-packet, the converse is not true; in this paper we find four examples of ABV-packets that are not Arthur packets. See Section 0.3 for more detail on this part of the examples.

(§0.4) When G admits an elliptic endoscopic group G' and an infinitesimal parameter $\lambda' : W_F \to {}^LG'$ such that $\lambda = \epsilon \circ \lambda'$ with $\epsilon : {}^LG' \to {}^LG$, we show how the transfer of stable distributions attached to Arthur parameter for G' to G may be apprehended through the restriction of equivariant perverse sheaves from V_λ to $V_{\lambda'}$. To see this, for each simple $\mathcal{P} \in \mathsf{Per}_{H_\lambda}(V_\lambda)$, we calculate every term in the identity

$$\mathrm{trace}_{a'_s} \mathsf{NEv}_{\psi'} \mathcal{P}|_{V'} = (-1)^{\dim C - \dim C'} \mathrm{trace}_{a_s} \mathsf{NEv}_\psi \mathcal{P}, \tag{6}$$

where $\psi' \in T^*_{C'}(V')_{\mathrm{reg}}$ with image $\psi \in T^*_C(V)_{\mathrm{reg}}$, where the semisimple $s \in \widehat{G}$ is part of the endoscopic data of G', a_s is the image of s in A_ψ and a'_s is the image of s in A'_ψ. See Section 0.4 for more detail on this part of the examples.

Although do not show every calculation in every example, in Section 0 we explain the ideas needed and then illustrate them as they apply appear in the examples.

We now describe the highlights of the six examples (38 admissible representations, 12 p-adic groups) in this paper.

(§1) In Section 1 we take $G = \mathrm{SL}(2)$ and consider the L-packet of quadratic unipotent representations of $\mathrm{SL}(2, F)$ when the residual characteristic of F is odd. We use this example to show how to reduce the problem of calculating Arthur packets and Arthur's transfer coefficients, geometrically, to the case of unipotent representations, using [7, Theorem 3.1.1]. There are three elliptic endoscopic groups relevant to this L-packet of admissible representations of $\mathrm{SL}(2, F)$, and we show how to apprehend transfer in each case using the geometric perspective. In this example we also explain how to extend [7] to the inner form of $\mathrm{SL}(2, F)$.

(§2) Because the representations in the first example are tempered, the geometry was degenerate. In Section 2 we take $G = \mathrm{SO}(3)$ split over F and choose a non-tempered unipotent representation $\pi(\phi_0)$ of $\mathrm{SO}(3, F)$. Then we find the 2 admissible representations of the anisotropic form G_1 of G that share the same infinitesimal parameter $\lambda : W_F \to {}^LG$ as $\pi(\phi_0)$. Even in this simple case the calculation of the vanishing cycles of simple objects in $\mathsf{Per}_{Z_{\widehat{G}}(\lambda)}(V_\lambda)$ is interesting. This example plays a role in Sections 3 and 6.

(§3) In Section 3 we take $G = \mathrm{PGL}(4)$ and choose a tamely ramified infinitesimal parameter $\lambda : W_F \to {}^LG$ such that its restriction to inertia has order $q + 1$. Again we use [7, Theorem 3.1.1] to reduce the problem of calculating the transfer coefficients, geometrically, to a unipotent representation of $\mathrm{PGL}(2)$ and then we use Section 2 to make the calculations. This example also illustrates a case when the map from pure rational forms to inner rational forms is not injective.

(§4) In Section 4 we return to unipotent representations of odd orthogonal groups and choose an infinitesimal parameter $\lambda : W_F \to {}^LG$ for $G = \mathrm{SO}(5)$ such that the image of Frobenius is regular semisimple. This example plays a role in Section 6.

(§5) Although endoscopy played a role in Section 1, the unipotent representations of pure rational forms of $G = \mathrm{SO}(5)$ treated in Section 5 give a more interesting illustration of how to apprehend endoscopy through equivariant restriction of perverse sheaves. In contrast to Section 4, in Section 5 the image of Frobenius under the unramified infinitesimal parameter $\lambda : W_F \to {}^LG$ is singular. This example shows some interesting new features in the calculation of vanishing cycles of perverse sheaves, and these play a role in Section 6. Here we find two examples of ABV-packets that are not Arthur packets.

(§6) Section 6 is the heart of this paper. Here we take $G = \mathrm{SO}(7)$ and consider unipotent representations with an infinitesimal parameter $\lambda : W_F \to {}^LG$ such that the image of Frobenius is singular. This rich example allows us to explore a wide range of interesting phenomena. We find 10 admissible representations of $G(F)$ with this infinitesimal parameter and a further 5 admissible representations of its pure rational form. One of these representations, denoted by $\pi(\phi_7, +-)$ in Section 6, is supercuspidal, a further 3 are tempered, and the remaining 11 are not tempered. We group these representations into L-packets and Arthur packets and find the stable and invariant distributions attached to Arthur parameters. Then, using vanishing cycles of perverse sheaves, we calculate all ABV-packets in this example using the functor Ev and verify that all Arthur packets are ABV-packets. We further verify (4). We also find two more ABV-packets that are not Arthur packets. The calculations of vanishing cycles of perverse sheaves in this section use some of the previously considered examples, but also involve some new work. We use this example to show that how the vanishing cycles of the Fourier transform of a perverse sheaf relates to the transpose of the vanishing cycles of the perverse sheaf. We verify the Kazhdan-Lusztig conjecture as it applies here and use this to confirm [7, Conjecture 3] in this example. Since the elliptic endoscopic group $G' = \mathrm{SO}(5) \times \mathrm{SO}(3)$ admits an infinitesimal parameter $\lambda' : W_F \to {}^LG'$ that factors λ, we show how Langlands-Shelstad transfer of stable distributions from G' to G may be calculated using equivariant restriction of perverse sheaves from V_λ to $V_{\lambda'}$.

Using techniques different from those employed in this paper (namely, microlocalisation of regular holonomic D-modules, rather than vanishing cycles of perverse sheaves) one of the authors of this paper has calculated many other examples of Adams-Barbasch-Vogan packets in his PhD thesis [19]. Specifically, if π is a unipotent representation of $\mathrm{PGL}(n)$, $\mathrm{SL}(n)$, $\mathrm{Sp}(2n)$ or $\mathrm{SO}(2n+1)$, of any of its pure rational forms, and if the image of Frobenius of the infinitesimal parameter of π is *regular* semisimple in the dual group, then all Adams-Barbasch-Vogan packets containing π have been calculated by finding the support of the microlocalisation of the relevant D-modules. This work overlaps with Sections 2 and 4, here. However, we found it difficult to calculate the finer properties of the microlocalisation of these D-modules required to determine the local transfer coefficients appearing in Arthur's work. This is one of the reasons we use vanishing cycles of perverse sheaves in [7] and in this paper.

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0. Template for the examples

Here, in Section 0, we explain the plan for all the examples. We have tried to make the examples (Sections 1 through 6) as brief as possible, by making repeated reference back to this section.

In each example we begin by choosing G from the following list of split algebraic groups over a p-adic field F: in order, we take G to be SL(2), SO(3), PGL(4), SO(5), SO(5) again, and finally, SO(7). In each case we find $Z^1(F,G)$, and thus all pure rational forms of G, and relate these to the inner forms of G using the maps

$$H^1(F,G) \to H^1(F,G_{\mathrm{ad}}) \to H^1(F,\mathrm{Aut}(G)).$$

Every pure rational form $\delta \in Z^1(F,G)$ determines a rational form G_δ of G, often also called a pure rational form of G. The examples that we consider illustrate the fact that the maps above are neither injective nor surjective, in general. In each case we also fix an infinitesimal parameter

$$\lambda : W_F \to {}^LG.$$

We consider two infinitesimal parameters λ for SO(5), but otherwise choose one λ for each group in the list, above.

Having fixed G and $\lambda : W_F \to {}^LG$, we consider the conjectures from [7]. Although we do prove these conjectures by brute force calculation in these examples, that was not our objective. Rather, our goal here is to show how to use results from [7] and [8] to calculate the stable distributions in Arthur's local result [3, Theorem 1.5.1] and also how to calculate the coefficients that appear when these stable distributions are transferred to certain endoscopic groups. As a consequence, we give complete examples of [3, Theorem 1.5.1], and explain how to use geometry to make the calculations.

0.1. **Arthur packets.**

We enumerate all admissible representations π of all pure rational forms δ of G with a shared infinitesimal parameter λ. We show how these representations fall into L-packets, indexed by Langlands parameters ϕ with infinitesimal parameter λ. Then if ϕ is of Arthur type, we find corresponding the Arthur packet. We find the stable distributions attached to these L-packets, and also all the invariant distributions obtained from these representations by endoscopy.

0.1.1. *Parameters.* We find all Langlands parameters $\phi : L_F \to {}^LG$ such that $\phi(w,d_w) = \lambda(w)$, where $d_w \in \mathrm{SL}(2)$ is defined by $d_w = \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})$. As in [7], we write $P_\lambda({}^LG)$ for these Langlands parameters and $\Phi_\lambda(G/F)$ for the isomorphism classes of these Langlands parameters under $Z_{\widehat{G}}(\lambda)$-conjugation.

Then we find all Arthur parameters $\psi : L_F \times \mathrm{SL}(2,\mathbb{C}) \to {}^LG$ such that $\psi(w,d_w,d_w) = \lambda(w)$. As in [7], the set of Arthur parameters that arise in this way is denoted by $Q_\lambda({}^LG)$. Although the map $Q_\lambda({}^LG) \to P_\lambda({}^LG)$ is injective, it is not surjective in general.

0.1.2. *Admissible representations and their pure L-packets.* Now we can list all representations (π, δ) of all pure rational forms of G, in the sense of [21], with infinitesimal parameter λ. This means that for every pure rational form $\delta \in Z^1(F, G)$, we find all irreducible admissible representations π of the rational form G_δ attached to G, such that the Langlands parameter ϕ for π lies in $P_\lambda({}^LG)$. These representations are not tempered, in most cases considered in this paper. When the pure rational form δ is clear from context, we may write π for (π, δ).

We arrange these admissible representations into L-packets and into pure L-packets. For this, we must find the component group

$$A_\phi := Z_{\widehat{G}}(\phi)/Z_{\widehat{G}}(\phi)^0,$$

for each $\phi \in P_\lambda({}^LG)$. According to the pure Langlands correspondence [21], equivalence classes of irreducible representations of pure rational forms of G with infinitesimal parameter λ are indexed by the set

$$\Xi_\lambda({}^LG) := \left\{(\phi, \rho) \mid \phi \in P_\lambda({}^LG)/Z_{\widehat{G}}(\lambda),\ \rho \in \mathrm{Irrep}(A_\phi)\right\}.$$

By abuse of notation, we write $\pi(\phi, \rho)$ for an irreducible admissible representation of $G(F)$ corresponding to a pair (ϕ, ρ) above. Each $\rho \in \mathrm{Irrep}(A_\phi)$ determines the class of a pure rational form, denoted by $\delta_\rho \in Z^1(F, G)$, so the L-packet for ϕ and a rational form G_δ is

$$\Pi_\phi(G_\delta(F)) = \{[\pi(\phi, \rho)] \mid \phi \in P_\lambda({}^LG),\ \rho \in \mathrm{Irrep}(A_\phi),\ [\delta_\rho] = [\delta] \in H^1(F, G)\}$$

We find these L-packets, for all $\phi \in P_\lambda({}^LG)$ and all $\delta \in Z^1(F, G)$, in our examples. We also find the pure L-packets:

$$\Pi_{\mathrm{pure},\phi}(G/F) = \{[\pi(\phi, \rho), \delta_\rho] \mid \phi \in P_\lambda({}^LG),\ \rho \in \mathrm{Irrep}(A_\phi)\},$$

for all $\phi \in P_\lambda({}^LG)$ To simplify notation slightly, we often write $\pi(\phi, \rho)$ for the pair $(\pi(\phi, \rho), \delta_\rho)$.

0.1.3. *Multiplicity matrix.* To describe the representations with infinitesimal parameter λ we present the multiplicity $m_{\mathrm{rep}}((\phi, \rho), (\phi', \rho'))$ of $\pi(\phi, \rho)$ in the standard module $M(\phi, \rho)$ so that in the Grothendieck group of admissible representations generated by $\Pi_{\mathrm{pure},\lambda}(G/F)$ we have

$$M(\phi', \rho') \equiv \sum_{(\phi,\rho)} m_{\mathrm{rep}}((\phi, \rho), (\phi', \rho'))\ \pi(\phi, \rho),$$

where the sum is taken over all $\phi \in P_\lambda({}^LG)$ and all $\rho \in \mathrm{Irrep}(A_\phi)$.

The idea of computing the multiplicities in the standard modules is to compare the Jacquet modules of the standard modules with those of irreducible representations. To be more precise, one can always make some guesses of what should be inside the standard modules by looking at the corresponding inducing representations. Then one can further argue that they are really there. To see there is nothing else, it is enough to show that the Jacquet modules of the standard modules have been exhausted by these representations. We give a sample calculation using this strategy in Section 6.1.3.

0.1.4. *Arthur packets.* Recall $Q_\lambda({}^LG)$ from Section 0.1.1. For each $\psi \in Q_\lambda({}^LG)$ we show how the admissible representations above are grouped into Arthur packets

$$\Pi_\psi(G_\delta(F))$$

for rational forms δ of G. Of course, $\Pi_\psi(G_\delta(F))$ contains the L-packet $\Pi_{\phi_\psi}(G_\delta(F))$; our interest is in the representations in $\Pi_\psi(G_\delta(F))$ that are not contained in $\Pi_{\phi_\psi}(G_\delta(F))$; we refer to these as *coronal representations* in [7]. In fact, we further recall the adaptation of Arthur packets to pure rational forms and find the pure Arthur packets

$$\Pi_{\mathrm{pure},\psi}(G/F)$$

themselves.

Arthur's main local result for quasisplit classical groups is expressed in terms of a map

$$\begin{array}{rcl} \Pi_\psi(G(F)) & \to & \widehat{\mathcal{S}_\psi}, \\ \pi & \mapsto & \langle \cdot\,, \pi \rangle_\psi \end{array} \tag{7}$$

where $\mathcal{S}_\psi = Z_{\widehat{G}}(\psi)/Z_{\widehat{G}}(\psi)^0\, Z(\widehat{G})^{\Gamma_F}$. As we saw in [7], this is easily rephrased in terms of a map

$$\Pi_\psi(G(F)) \to \mathrm{Irrep}(A_\psi), \tag{8}$$

where

$$A_\psi = Z_{\widehat{G}}(\psi)/Z_{\widehat{G}}(\psi)^0.$$

We find this map in our examples. In fact, using [3, Inner twists], we find the conjectured extension

$$\Pi_{\mathrm{pure},\psi}(G/F) \to \mathrm{Irrep}(A_\psi) \tag{9}$$

which includes the non-quasi-split pure rational forms of G, as discussed in [7].

0.1.5. *Aubert duality.* Aubert involution preserves the infinitesimal parameter λ and so defines an involution on $\mathsf{K}\Pi_\lambda(G_\delta(F))$, for every pure ration form δ for G. For $\pi \in \Pi_\lambda(G_\delta(F))$ we use the notation $\hat{\pi}$ for the admissible representation such that $(-1)^{a(\pi)}\hat{\pi}$ is the Aubert dual of π in $\mathsf{K}\Pi_\lambda(G_\delta(F))$. When restricted to Arthur packets, Aubert duality defines a bijection

$$\begin{array}{rcl} \Pi_\psi(G_\delta(F)) & \to & \Pi_{\hat{\psi}}(G_\delta(F)) \\ \pi & \mapsto & \hat{\pi}, \end{array}$$

where $\hat{\psi}(w,x,y) := \psi(w,y,x)$. We display this bijection in our examples.

Although the component groups A_ψ and $A_{\hat{\psi}}$ are isomorphic, a comparison of the characters $\langle\,\cdot\,,\pi\rangle_\psi$ and $\langle\,\cdot\,,\hat{\pi}\rangle_{\hat{\psi}}$ shows that they do not coincide, in general. Accordingly, their ratio defines a character χ_ψ of A_ψ such that

$$\langle s, \hat{\pi}\rangle_{\hat{\psi}} = \chi_\psi(s)\langle s, \pi\rangle_\psi, \tag{10}$$

for $s \in Z_{\widehat{G}}(\psi)$ where, as usual, we use the map $Z_{\widehat{G}}(\psi) \to A_\psi$. Our examples show that this character χ_ψ of A_ψ is given by

$$\chi_\psi = \epsilon_\psi^{M/W}\epsilon_{\hat{\psi}}^{M/W}, \tag{11}$$

where $\epsilon_\psi^{M/W}$ is the character of A_ψ appearing in [22, Theorem 8.9]. As explained in [23, Introduction], the character $\epsilon_\psi^{M/W}$ measures the difference between Mœglin's parametrization of representations in Π_ψ by A_ψ and Arthur's parametrization of representations in Π_ψ by A_ψ. We compute the character χ_ψ in our examples; it is non-trivial in Sections 5.1.5 and 6.1.5 only.

0.1.6. *Stable distributions and endoscopy.* Armed with (8), we easily find the coefficients in the stable invariant distribution

$$\Theta_\psi^G = \sum_{\pi \in \Pi_\psi(G(F))} \langle s_\psi, \pi \rangle_\psi \operatorname{trace} \pi, \tag{12}$$

where s_ψ denotes the image of the non-trivial central element in SL(2) in A_ψ. Likewise, for $s \in Z_{\widehat{G}}(\psi)$ we compute

$$\Theta_{\psi,s}^G = \sum_{\pi \in \Pi_\psi(G(F))} \langle s s_\psi, \pi \rangle_\psi \operatorname{trace} \pi. \tag{13}$$

Arthur's work shows that $\Theta_{\psi,s}$ is the Langlands-Shelstad transfer of the invariant distribution

$$\Theta_{\psi'}^{G'} = \sum_{\pi' \in \Pi_{\psi'}(G'(F))} \langle s_{\psi'}, \pi' \rangle_{\psi'} \operatorname{trace} \pi', \tag{14}$$

attached to an endoscopic group G' attached to s and where $\psi' : L_F \times \mathrm{SL}(2) \to {}^L G'$ factors through ${}^L G' \to {}^L G$, defining $\psi' : L_F \times \mathrm{SL}(2) \to {}^L G'$. For use below, we illustrate this fact in our examples by choosing a particular $s \in \widehat{G}$ and computing $\Theta_{\psi'}$.

In order to illuminate [7, Conjecture 2] we use (9) to exhibit the virtual representations

$$\eta_\psi = \sum_{[\pi,\delta] \in \Pi_{\mathrm{pure},\psi}(G/F)} e(\delta) \langle s_\psi, [\pi, \delta] \rangle_\psi \; [\pi, \delta] \tag{15}$$

and

$$\eta_{\psi,s} = \sum_{[\pi,\delta] \in \Pi_{\mathrm{pure},\psi}(G/F)} e(\delta) \langle s s_\psi, [\pi, \delta] \rangle_\psi \; [\pi, \delta] \tag{16}$$

for $s \in Z_{\widehat{G}}(\psi)$, as defined in [7]. Likewise we find

$$\eta_{\psi'} = \sum_{[\pi',\delta'] \in \Pi_{\mathrm{pure},\psi'}(G'/F)} e(\delta') \langle s_{\psi'}, [\pi', \delta'] \rangle_{\psi'} \; [\pi', \delta'] \tag{17}$$

with s and ψ' as above.

0.2. **Vanishing cycles of perverse sheaves.** Having reviewed Arthur packets and transfer coefficients for the chosen G and $\lambda : W_F \to {}^L G$, we now turn to geometry. In this section we introduce the geometric tools needed to demonstrate [7, Conjecture 2] and calculate the coefficients $\langle s s_\psi, [\pi, \delta] \rangle_\psi$ appearing above. This is done by a brute force calculation of the exact functor

$${}^p\mathsf{Ev} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(T_H^*(V)_{\mathrm{reg}}),$$

defined in [7, Section 5], on simple objects, following a strategy that we now explain.

0.2.1. *Vogan variety.* We find the variety $V := V_\lambda$ attached to the infinitesimal parameter $\lambda : W_F \to {}^LG$, the action of $H := H_\lambda := Z_{\widehat{G}}(\lambda)$ on V, and the stratification of V into H-orbits. If λ is not unramified, we use [7, Theorem 3.1.1] to replace the action $H_\lambda \times V_\lambda \to V_\lambda$ with $H_{\lambda_{\mathrm{nr}}} \times V_{\lambda_{\mathrm{nr}}} \to V_{\lambda_{\mathrm{nr}}}$ where $\lambda_{\mathrm{nr}} : W_F \to {}^LG_\lambda$ is the "unramification" of $\lambda : W_F \to {}^LG$. We may now assume λ is unramified and $\lambda(\mathrm{Fr})$ is elliptic semisimple in $\widehat{G}$.

For classical groups, the variety V admits a description which is quite convenient for calculations, as we now explain. If the type of G is A_n, the variety V can be decomposed as a finite direct product of varieties according to

$$V \cong \mathrm{Hom}(E_0, E_1) \times \mathrm{Hom}(E_1, E_2) \times \cdots \mathrm{Hom}(E_{r-1}, E_r),$$

where each E_i is an eigenspace for $\lambda(\mathrm{Fr})$ with eigenvalue λ_i. We may then denote elements of V, *i.e.*, quiver representations, by $v = (v_{i,i+1})_i$, for $v_{i,i+1} \in \mathrm{Hom}(E_i, E_{i+1})$. Then

$$H \cong \mathrm{GL}(E_0) \times \mathrm{GL}(E_1) \times \cdots \times \mathrm{GL}(E_r)$$

acting on $\mathrm{Hom}(E_0, E_1) \times \mathrm{Hom}(E_1, E_2) \times \cdots \mathrm{Hom}(E_{r-1}, E_r)$ by $h_i \cdot v_{i,i+1} = v_{i,i+1} \circ h_i^{-1}$ and $h_i \cdot v_{i-1,i} = h_i \circ v_{i-1,i}$ and $h_i \cdot v_{j,j+1} = v_{j,j+1}$ for $j \neq i, i-1$. The H-orbit of $v \in V$ is fully characterized by the collection of integers

$$r_{ij} := \mathrm{rank}(v_{j-1,j} \circ \cdots \circ v_{i,i+1}).$$

One derives a natural set of inequalities which describes admissible collections of ranks. The partial order of adjacency is identical to the partial ordering on the symbols $(r_{ij})_{ij}$.

Passing from the case when the derived group of G is of type A_n to any of B_n, C_n or D_n simply results in an identification of the λ_i eigenspace of $\lambda(\mathrm{Fr})$ with the dual of the λ_i^{-1} eigenspace. There are essentially two cases to consider: either $E_i = E_{r-i}^*$. or no two of $E_0, \ldots, E_r$ are dual. In the later case, V is isomorphic to one arising from an inclusion of a subgroup of type A_n and one can freely study the variety by passing to this subgroup. In the former case, there are essentially four sub-cases depending on if we are inside an orthogonal or symplectic group and if r is even or odd. In either case the variety we are studying is the one where $v_{i,i+1} = v_{r-i-1,r-i}^t$ and the group acting factors through $h_i = h_{r-i}^t$. These equations impose further, obvious, restrictions on the set of admissible collections of ranks/nullities, but otherwise the collection of strata is still indexed by the set of admissible vectors $(r_{ij})_{ij}$ and the adjacency relations do not change.

For simplicity of exposition one can describe these varieties which occur when G is of type B_n as one of

$$\mathrm{Hom}(E_0, E_1) \times \mathrm{Hom}(E_1, E_2) \times \cdots \mathrm{Hom}(E_{\ell-1}, E_\ell) \times \mathrm{Sym}^2(E_\ell^*)$$

with the group acting being $\mathrm{GL}(E_i)$ at every factor or

$$\mathrm{Hom}(E_0, E_1) \times \mathrm{Hom}(E_1, E_2) \times \cdots \mathrm{Hom}(E_{\ell-1}, E_\ell)$$

Where the group acts by $\mathrm{GL}(E_i)$ on every factor except E_ℓ where the group is $\mathrm{Sp}(E_\ell)$. When G is of type C_n or D_n they are

$$\mathrm{Hom}(E_0, E_1) \times \mathrm{Hom}(E_1, E_2) \times \cdots \mathrm{Hom}(E_{\ell-1}, E_\ell) \times \mathrm{Alt}^2(E_\ell^*)$$

with the group acting being $\mathrm{GL}(E_i)$ at every factor.

$$\mathrm{Hom}(E_0, E_1) \times \mathrm{Hom}(E_1, E_2) \times \cdots \mathrm{Hom}(E_{\ell-1}, E_\ell),$$

where the group acts by $\mathrm{GL}(E_i)$ on every factor except E_ℓ where the group is $\mathrm{O}(E_\ell)$. In all of these cases, ℓ is either $r/2$ or $(r+1)/2$, and the combinatorial data which describes the strata is still the collection of ranks $r_{i,j}$ for $0 \leq i < j \leq r$.

0.2.2. *Orbit duality.* The cotangent bundle $T^*(V)$ is equipped with two important functions, used extensively in [7]: the natural pairing $(\cdot\,|\,\cdot) : T^*(V) \to \mathbb{A}^1$ which coincides with the restriction of the Killing form on $\mathfrak{j}_\lambda$; and $[\,\cdot\,,\,\cdot\,] : T^*(V) \to \mathfrak{h}$ which coincides with the restriction of the Lie bracket on $\mathfrak{j}_\lambda$. In particular, for every H-orbit C in V,

$$T^*_C(V) = \{(x,\xi) \in T^*(V) \mid x \in C,\ [x,\xi] = 0\}.$$

In the examples, we present the duality between H-orbits C in V and H-orbits C^* in V^*, defined by the property that they have isomorphic conormal bundles

$$\overline{T^*_C(V)} \cong \overline{T^*_{C^*}(V^*)}$$

under $T^*(V) \to T^*(V^*)$ given by $(x,\xi) \mapsto (\xi, x)$, where we identify V^{**} with V using $(\cdot\,|\,\cdot)$. In fact, this duality between H-orbits in V and H-orbits in V^* is also characterized by the following statement:

$$T^*_C(V)_{\mathrm{reg}} \subseteq C \times C^*, \tag{18}$$

where

$$T^*_C(V)_{\mathrm{reg}} := T^*_C(V) \setminus \bigcup_{C \subsetneq \overline{C'}} \overline{T^*_{C'}(V)}.$$

In the examples we present all this information by describing the conormal bundle

$$T^*_H(V) := \bigcup_C T^*_C(V),$$

where the union is taken over all H-orbits C in V and the union is taken in $T^*(V)$. We also describe the regular conormal bundle

$$T^*_H(V)_{\mathrm{reg}} := \bigcup_C T^*_C(V)_{\mathrm{reg}}.$$

From this, one simply restricts the bundle maps $T^*(V) \to V$ and $T^*(V) \to V^*$ to $T^*_C(V)_{\mathrm{reg}}$ to recover C and its dual orbit C^*.

0.2.3. *Equivariant perverse sheaves.* The next step is to find all simple objects in category $\mathsf{Per}_H(V)$ of H-equivariant perverse sheaves on V. Again, we use [7, Theorem 3.1.1] to reduce to the case when λ is unramified and $\lambda(\mathrm{Fr})$ is hyperbolic.

It is convenient to begin by enumerating all equivariant local systems $\mathcal{L}$ on all H-orbits C in V. This is done by picking a base point $x \in C$ and computing the equivariant fundamental group

$$A_x := \pi_0(Z_H(x)) \cong \pi_1(C,x)_{Z_H(x)^0}.$$

Since the isomorphism type of this group is independent of the choice of base point, this group is commonly denoted by A_C. For the groups G that we consider here, the fundamental group A_C is always abelian, but this is not true in general. In any case, the choice of $x \in C$ determines an equivalence

$$\mathsf{Rep}(A_C) \to \mathsf{Loc}_H(C).$$

It is now easy to enumerate all simple objects in category $\mathsf{Per}_H(V)$:

$$\mathsf{Per}_H(V)^{\mathrm{simple}}_{/\mathrm{iso}} = \left\{ \mathcal{IC}(C,\mathcal{L}) \mid H\text{-orbit } C \subseteq V,\ \mathcal{L} \in \mathsf{Loc}_H(V)^{\mathrm{simple}}_{/\mathrm{iso}} \right\}.$$

We will need to compute the equivariant perverse sheaves $\mathcal{IC}(C,\mathcal{L})$ themselves, or rather, their image in the Grothendieck group

$$\mathsf{Per}_H(V) \to \mathsf{KPer}_H(V) = \mathsf{KD}^b_{c,H}(V).$$

For every H-orbit C in V and every H-equivariant local system $\mathcal{L}$ on V, consider the shifted standard sheaf

$$\mathcal{S}(C,\mathcal{L}) := j_{C!}\ \mathcal{L}[\dim C],$$

where $j_C : C \hookrightarrow V_\lambda$ is inclusion. Then, in $\mathsf{KPer}_{H_\lambda}(V_\lambda)$ we have

$$\mathcal{IC}(C,\mathcal{L}) \equiv \sum_{(C',\mathcal{L}')} m_{\mathrm{geo}}((C',\mathcal{L}'),(C,\mathcal{L}))\ \mathcal{S}(C',\mathcal{L}')$$

and $m_{\mathrm{geo}}((C,\mathcal{L}),(C,\mathcal{L})) = 1$ and $m_{\mathrm{geo}}((C',\mathcal{L}'),(C,\mathcal{L})) = 0$ unless $C' \leq C$. We refer to the matrix m_{geo} as the geometric multiplicity matrix. Set

$$\mathcal{L}^\sharp := \mathcal{IC}(C,\mathcal{L})[-\dim C] \qquad \text{and} \qquad \mathcal{L}^\natural := \mathcal{S}(C,\mathcal{L})[-\dim C].$$

Then, in $\mathsf{KPer}_{H_\lambda}(V_\lambda)$,

$$\mathcal{L}^\sharp \equiv \sum_{(C',\mathcal{L}')} (-1)^{\dim C - \dim C'} m_{\mathrm{geo}}((C',\mathcal{L}'),(C,\mathcal{L}))\ \mathcal{L}'^\natural.$$

The purity result of Lusztig shows that $\mathcal{L}^\sharp$ is cohomologically concentrated in even degrees, so

$$m'_{\mathrm{geo}}((C',\mathcal{L}'),(C,\mathcal{L})) := (-1)^{\dim C - \dim C'} m_{\mathrm{geo}}((C',\mathcal{L}'),(C,\mathcal{L}))$$

is a non-negative integer. We refer to the matrix m'_{geo} as the normalised geometric multiplicity matrix.

We compute the normalised geometric multiplicity matrix m'_{geo} in each example in this paper. In Sections 1.2.2 and 3.2.2 we use [7, Theorem 3.1.1] to make this calculation. In Sections 2.2.2, 4.2.2, 5.2.3 and 6.2.3 we give examples of the following strategy. For each stratum $C \subseteq V$ and each local system $\mathcal{L}$ on C, we construct a proper cover $\pi : \widetilde{C} \to \overline{C}$ such that $\widetilde{C}$ is smooth and $\mathcal{IC}(C,\mathcal{L})$ appears in $\pi_! \mathbb{1}_{\widetilde{C}}[\dim \widetilde{C}]$. We can explicitly describe the fibres of π over each stratum in $\overline{C}$ and typically arrange things so that the cover is semi-small, though this is not essential. We then find all the other simple perverse sheaves $\mathcal{IC}(C',\mathcal{L}')$, for $C' \leq C$, appearing in $\pi_! \mathbb{1}_{\widetilde{C}}[\dim \widetilde{C}]$, using the Decomposition Theorem. By doing this for C and all strata on the boundary of C, we can describe $\mathcal{IC}(C,\mathcal{L})$. Note that this process is performed inductively on $\dim C$, as well as on $\mathrm{rank}(\pi_! \mathbb{1}_{\widetilde{C}})|_C$.

0.2.4. *Cuspidal support decomposition and Fourier transform.* Category $\mathsf{Per}_H(V)$ decomposes into a direct sum of full subcategories indexed by cuspidal pairs for $\widehat{G}$, or more correctly, cuspidal local systems on cuspidal pairs [14, Proposition 8.16]. We refer to this as the *cuspidal support decomposition* of $\mathsf{Per}_H(V)$:

$$\mathsf{Per}_H(V) = \bigoplus_{(L,\mathcal{O},\mathcal{E})} \mathsf{Per}_H(V)_{L,C,\mathcal{E}},$$

where the sum is taken over all cuspidal Levi subgroups L of $\widehat{G}$, and all cuspidal local systems $\mathcal{E}$ on nilpotent orbits $\mathcal{O} \subset \mathrm{Lie}\, L$, up to $\widehat{G}$-conjugation. In the cases we consider there is only one $(\mathcal{O},\mathcal{E})$ for every cuspidal Levi L, so we abbreviate $\mathsf{Per}_H(V)_{L,C,\mathcal{E}}$ to $\mathsf{Per}_H(V)_L$. In each example we partition the simple objects in $\mathsf{Per}_H(V)$ according to this decomposition. Simple objects in $\mathsf{Per}_H(V)_L$ are characterized by the property that they appear in the semisimple complex formed by parabolic induction along Vogan varieties from the cuspidal local system on $\mathrm{Lie}\, L \cap V$; see [15].

The cuspidal support decomposition of $\mathsf{Per}_H(V)$ offers insight into the blocks that appearing within the geometric multiplicity matrix. It is also quite helpful for finding the proper covers appearing in Section 0.2.3.

We also compute the Fourier transform

$$\mathsf{Ft} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(V^*)$$

on all simple objects. This functor is compatible with the cuspidal support decomposition in the sense that Ft restricts to $\mathsf{Per}_H(V)_L \to \mathsf{Per}_H(V^*)_L$.

0.2.5. *Local systems on the regular conormal bundle.* In preparation for the calculation of $\mathsf{Ev} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(T^*_H(V)_{\mathrm{reg}})$, we must describe local systems on H-orbits $T^*_C(V)_{\mathrm{sreg}}$ and also show how local systems relate to the pullback of local systems along the bundle maps $T^*_C(V)_{\mathrm{sreg}} \to C$ and $T^*_C(V)_{\mathrm{sreg}} \to C^*$. For this we pick a base point $(x,\xi) \in T^*_C(V_\lambda)_{\mathrm{sreg}}$ and compute the equivariant fundamental groups

$$A_{(x,\xi)} = \pi_0(Z_H(x,\xi))) = \pi_1(T^*_C(V)_{\mathrm{sreg}}, (x,\xi))_{Z_H(x,\xi)^0}.$$

The isomorphism type of $A_{(x,\xi)}$ is independent of the choice of base point; it is precisely the microlocal fundamental group of C, denoted by A^{mic}_C. So the choice of base point determines an equivalence

$$\mathsf{Rep}(A^{\mathrm{mic}}_C) \to \mathsf{Loc}_H(T^*_C(V)_{\mathrm{sreg}}).$$

We use this to enumerate the simple objects in $\mathsf{Loc}_H(T^*_C(V)_{\mathrm{sreg}})$ and then to describe the functors

$$\mathsf{Loc}_H(C) \longrightarrow \mathsf{Loc}_H(T^*_C(V)_{\mathrm{sreg}}) \longleftarrow \mathsf{Loc}_H(C^*)$$

obtained by pullback the along the projections

$$C \longleftarrow T^*_C(V)_{\mathrm{sreg}} \longrightarrow C^*,$$

by way of the induced homomorphisms of equivariant fundamental groups.

$$A_x \longleftarrow A_{(x,\xi)} \longrightarrow A_\xi.$$

0.2.6. *Vanishing cycles of perverse sheaves.* Here we present the results of applying the functor

$${}^p\mathsf{Ev} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(T^*_H(V)_{\mathrm{sreg}})$$

to simple objects in $\mathsf{Per}_H(V)$. Recall from [7, Section 5] that ${}^p\mathsf{Ev} = \oplus_{C'} {}^p\mathsf{Ev}_{C'}$, where

$${}^p\mathsf{Ev}_{C'} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(T^*_{C'}(V)_{\mathrm{reg}}).$$

is defined by

$${}^p\mathsf{Ev}_C(\mathcal{F}) = \mathsf{R}\Phi_{(\cdot|\cdot)}[-1](\mathcal{F} \boxtimes \mathbb{1}_{C^*})|_{T^*_C(V)_{\mathrm{reg}}}[\dim C^*],$$

where $(\cdot|\cdot) : T^*(V) \to \mathbb{A}^1$ appeared in Section 0.2.2. Recall also from [7, Section 5] that

$$({}^p\mathsf{Ev}\,\mathcal{F})_{(x,\xi)} = (\mathsf{R}\Phi_\xi[-1]\mathcal{F})_x[\dim C^*],$$

for all $(x,\xi) \in T^*_H(V)_{\mathrm{reg}}$.

We present the results of our calculations in a table which offers two perspectives on ${}^p\mathsf{Ev}$. Then recall that if $\mathcal{IC}(C,\mathcal{L})$ is simple, then ${}^p\mathsf{Ev}_{C'}\,\mathcal{IC}(C,\mathcal{L})[-\dim V]$ is a local system on $T^*_{C'}(V)_{\mathrm{reg}}$ and this local system is determined by its restriction $\mathsf{Evs}_{C'}\,\mathcal{IC}(C,\mathcal{L})$ to the H-orbit $T^*_{C'}(V)_{\mathrm{sreg}}$. Our tables record ${}^p\mathsf{Ev}\,\mathcal{IC}(C,\mathcal{L})$ in form $\oplus_{C'}\mathcal{IC}(\mathcal{O}',\mathcal{E}')$, where $\mathcal{O}' := T^*_{C'}(V)_{\mathrm{sreg}}$. To describe each $\mathcal{E}'$, we use the base points $(x',\xi') \in T^*_{C'}(V)_{\mathrm{sreg}}$ to view $\mathsf{Evs}_{C'}\,\mathcal{IC}(C,\mathcal{L})$ as a representation of the equivariant fundamental group $A_{(x',\xi')}$ of

$T^*_{C'}(V)_{\mathrm{sreg}}$. The second part of the table records the characters of the representations $\mathsf{Ev}_{(x',\xi')}\,\mathcal{IC}(C,\mathcal{L})$ of $A_{(x',\xi')}$, as C' ranges over all strata in V and as $\mathcal{IC}(C,\mathcal{L})$ ranges over all simple objects in $\mathsf{Per}_H(V)$.

By [7, Theorem 5.8.1], we know that ${}^p\mathsf{Ev}_{C'}\,\mathcal{P} = 0$ unless $C' \subseteq \operatorname{supp}\mathcal{IC}(C,\mathcal{L})$, which is to say, unless $C' \leq C$. Moreover, [7, Theorem 5.8.1] also shows that in the case $C' = C$, we get

$$\mathsf{Evs}_C\,\mathcal{IC}(C,\mathcal{L}) = (p^*\mathcal{L})|_{T^*_C(V)_{\mathrm{sreg}}},$$

where $p : T^*_C(V) \to C$ is the restriction of the bundle map $T^*(V) \to V$. The local systems $(p^*\mathcal{L})|_{T^*_C(V)_{\mathrm{sreg}}}$ were described in Section 0.2.5 and they are worked out in the corresponding sections in each example. The work that remains to calculate ${}^p\mathsf{Ev}\,\mathcal{IC}(C,\mathcal{L})$, therefore, is the cases $\mathsf{Ev}_{C'}\,\mathcal{IC}(C,\mathcal{L})$ for $C' < C$.

To calculate ${}^p\mathsf{Ev}_{C'}\,\mathcal{IC}(C,\mathcal{L})$ for $C' < C$ we use [7, Lemma 5.4.2]. We describe our method in some detail here. From Section 0.2.3 we recall a proper map $\pi : \widetilde{C} \to \overline{C}$ from a smooth variety $\widetilde{C}$ chosen so that $\mathcal{IC}(C,\mathcal{L})$ appears in $\pi_!\mathbb{1}_{\widetilde{C}}[\dim\widetilde{C}]$. Using proper base change and the exactness of the vanishing cycles functor [7, Theorem 5.8.1] we find $\mathsf{Ev}_{C'}\,\mathcal{IC}(C,\mathcal{L})$ by computing

$$\left(\pi''_{s\,!}\ \mathsf{R}\Phi_{(\cdot\,|\,\cdot)\circ(\pi\times\mathrm{id}_{C'^*})}(\mathbb{1}_{\widetilde{C}\times C'^*})\right)|_{T^*_{C'}(V)_{\mathrm{reg}}}, \tag{19}$$

where π''_s is defined in [7, Section 5.2]. Since $\widetilde{C}\times C'^*$ is smooth and $\mathbb{1}_{\widetilde{C}\times C'^*}$ is a local system, the vanishing cycles

$$\mathsf{R}\Phi_{(\cdot\,|\,\cdot)\circ(\pi\times\mathrm{id}_{C'*})}(\mathbb{1}_{\widetilde{C}\times C'^*}) \tag{20}$$

is a skyscraper sheaf on the singular locus of $(\cdot\,|\,\cdot)\circ(\pi\times\mathrm{id}_{C'^*})$ on $\widetilde{C}\times C'^*$. This singular locus is easy to find using the Jacobian condition for smoothness,because of the explicit nature of π and because we have already found equations for $\overline{C'^*}$ in V^*. The map π''_s restricts to a proper map from this singular locus onto $T^*_C(V)$. In fact, this map is finite over $T^*_{C'}(V)_{\mathrm{reg}}$; this is a post-hoc consequence of the fact that the fibres of π''_s are closed and the stalks of the vanishing cycles functor are concentrated in a single degree. After restricting (20) to the preimage of $T^*_{C'}(V)_{\mathrm{reg}}$ under $\pi\times\mathrm{id}_{C'^*}$, we use the Decomposition Theorem to explicitly describe (19). While it is typically very easy to compute the rank of the resulting local system, determining the representation of the fundamental group that describes the local system is considerably more subtle as it depends on the local structure of the singularities. We give examples of these calculations in Sections 2.2.5, 4.2.5, 5.2.5 and 6.2.6. In Section 6.2.6 we give a sample calculation showing how the Lefschetz fixed-point formula may be used to make these calculations.

We observe that many of these calculations may be simplified considerably by a judicious use of the formula (**??**) from Section 0.2.8 and formula (34) from Section 0.4.

0.2.7. *Normalization of Ev and the twisting local system.* Having calculated ${}^p\mathsf{Ev} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(T^*_H(V)_{\mathrm{sreg}})$ in Section 0.2.6 we calculate the normalization of Ev, as given in [7, Section 5] by

$$\mathsf{NEv}_C := (\mathsf{Ev}_C\,\mathcal{IC}(C))^\vee \otimes \mathsf{Ev}_C : \mathsf{Per}_H(V) \to \mathsf{Loc}_H(T^*_C(V)_{\mathrm{reg}})$$

In the process, we explicit the rank-one local system $\mathcal{T}$ on $T^*_H(V)_{\mathrm{sreg}}$ given by

$$\mathcal{T}|_{T^*_C(V)_{\mathrm{sreg}}} = \mathsf{Evs}_C\,\mathcal{IC}(C).$$

0.2.8. *Fourier transform and vanishing cycles.* In this section we illustrate a remarkable formula showing how the Fourier transform interacts with vanishing cycles, or more precisely, with the functor Ev and its dual $\mathsf{Ev}^* : \mathsf{Per}_H(V^*) \to \mathsf{Per}_H(T^*_H(V^*)_{\mathrm{reg}})$ defined exactly as above but with V replaced by V^*:

$$a_* \, {}^p\mathsf{NEv} = {}^p\mathsf{Ev}^* \, \mathsf{Ft}, \tag{21}$$

where ${}^p\mathsf{NEv} = \mathcal{T}^\vee \otimes {}^p\mathsf{Ev}$ and $a : T^*(V) \to T^*(V^*)$ is the isomorphism $a(x,\xi) = (\xi, -x)$ where we identify the dual of V^* with V using $(\,\cdot\,|\,\cdot\,)$. We remark that (21) is equivalent to

$$a_* \, {}^p\mathsf{NEv}_C = {}^p\mathsf{Ev}_{C^*} \, \mathsf{Ft},$$

for all strata $C \subseteq V$.

In this paper we verify (21) by evaluating both sides on simple objects in $\mathsf{Per}_H(V)$ in each of our examples. The rank-one local system $\mathcal{T}$ is non-trivial in Sections 5.2.7 and 6.2.8.

0.2.9. *Arthur sheaves.* In the examples we close each version of Section 0.2 by displaying the Arthur sheaves $\mathcal{A}_C$ that appeared in [7, Section 6], for each stratum $C \subseteq V$. These equivariant perverse sheaves are defined, up to isomorphism, by

$$\mathcal{A}_C := \sum_{\mathcal{P} \in \mathsf{Per}_H(V)^{\mathrm{simple}}_{/\mathrm{iso}}} (\operatorname{rank} \mathsf{Ev}_C \, \mathcal{P}) \; \mathcal{P}$$

We also remark that

$$\mathsf{Ft} \, \mathcal{A}_C = \mathcal{A}_{C^*}. \tag{22}$$

0.3. **Adams-Barbash-Vogan packets.** Having calculated the vanishing cycles of perverse sheaves on Vogan varieties in Section 0.2, it is a simple matter now to find the Adams-Barbash-Vogan packets for all Langlands parameters with given infinitesimal parameter. In this section we also see that the Arthur packets described in the examples are indeed ABV-packets. But the real object of the conjectures from [7] are the characters $\langle\,\cdot\,,\pi\rangle_\psi$ of A_ψ that appear in Arthur's main local result, and their generalisations to pure rational forms of G. In this section we show

$$\langle s, \pi \rangle_\psi = \operatorname{trace}_{a_s} \mathsf{NEv}_{C_\psi} \, \mathcal{P}(\pi)$$

for $s \in Z_{\widehat{G}}(\psi)$ with image $a_s \in A_\psi$, and verify [7, Conjecture 1], [7, Conjecture 2] and [7, Conjecture 3].

0.3.1. *Admissible representations versus equivariant perverse sheaves.* As explained in [7], every Langlands parameter $\phi \in P_\lambda({}^LG)$ determines a point $x_\phi \in V$ and every $x \in V$ arises in this way. The function $\phi \mapsto x_\phi$ is also H-equivariant, so it induces a bijection between $\Phi_\lambda({}^LG)$ and the set of H-orbits in V. We write C_ϕ for the H-orbit of x_ϕ. There is a canonical isomorphism of groups

$$A_\phi \cong A_{C_\phi}, \tag{23}$$

where $A_\phi = \pi_0(Z_{\widehat{G}}(\phi))$ is the component group appearing in the pure Langlands correspondence. Consequently, there is a natural bijection between pairs (ϕ, ρ), where ρ is a representation of A_ϕ, and pairs $(C_\phi, \mathcal{L}_\rho)$, where $\mathcal{L}_\rho$ is the equivariant local system matching ρ under the isomorphism above. This, in turn, determines a bijection

$$\begin{array}{rcl} \Pi_{\mathrm{pure},\lambda}(G/F) & \to & \mathsf{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{/\mathrm{iso}} \\ (\pi, \delta) & \mapsto & \mathcal{P}(\pi, \delta) \end{array} \tag{24}$$

0.3.2. *ABV-packets.* Using this bijection, we determine the ABV-packets for all Langlands parameters with infinitesimal parameter λ, in each example, using the definition

$$\Pi^{\mathrm{ABV}}_{\mathrm{pure},\phi}(G/F) := \{[\pi,\delta] \in \Pi_{\mathrm{pure},\lambda}(G/F) \mid \mathsf{Ev}_{C_\phi}\, \mathcal{P}(\pi,\delta) \neq 0\}. \tag{25}$$

By restricting our attention to Langlands parameters of Arthur type, we readily verify that all Arthur packets for all admissible representations with infinitesimal parameter λ are ABV-packets:

$$\Pi^{\mathrm{ABV}}_{\mathrm{pure},\phi_\psi}(G/F) = \Pi_{\mathrm{pure},\psi}(G/F). \tag{26}$$

Having verified (26) in the examples, we turn to [7, Conjecture 2], which begins with the canonical isomorphism

$$A_\psi \cong A^{\mathrm{mic}}_{C_\psi},$$

where ψ is an Arthur parameter and where $C_\psi := C_{\phi_\psi}$. Right away, this isomorphism tells us that the character $\langle\, \cdot\, , \pi\rangle_\psi$ of A_ψ appearing in Arthur's main local result may be interpreted as an equivariant local system on $T^*_{C_\psi}(V)_{\mathrm{sreg}}$. How does the admissible representation π of $G(F)$ determine that local system? That question is answered by [7, Conjecture 2]: for every $s \in Z_{\widehat{G}}(\psi)$ and for every admissible representation π of $G(F)$,

$$\langle s s_\psi, \pi\rangle_\psi = (-1)^{\dim C_\psi - \dim C_\pi}\, \mathrm{trace}_{a_s}\, \mathsf{NEv}_{C_\psi}\, \mathcal{P}(\pi), \tag{27}$$

where a_s is the image of $s \in Z_{\widehat{G}}(\psi)$ in A_ψ and where C_π is the stratum in V attached to the Langlands parameter of π. In other words, the equivariant local system on $T^*_{C_\psi}(V)_{\mathrm{sreg}}$ determined by the admissible representation π of $G(F)$ is $\mathsf{NEv}_{C_\psi}\, \mathcal{P}(\pi)$.

Having calculated the left-hand side of (27) in Section 0.1 and right-hand side in Section 0.2, we can prove [7, Conjecture 2] in our examples by simply comparing the results of those calculations. In fact we confirm more in the examples, by showing that

$$\eta_{\psi,s} = \eta^{\mathsf{NEv}}_{\psi,s}, \tag{28}$$

for every Arthur parameter ψ with infinitesimal parameter λ and for every $s \in Z_{\widehat{G}}(\psi)$. Here, $\eta^{\mathsf{NEv}}_{\psi,s}$ is defined in [7, Section 6]:

$$\eta^{\mathsf{NEv}}_{\psi,s} = \sum_{[\pi,\delta]\in\Pi_{\mathrm{pure},\lambda}(G/F)} e(\delta)(-1)^{\dim C_\psi - C_\pi}\, \mathrm{trace}_{a_s}\, \mathsf{Ev}_{C_\psi}\, \mathcal{P}(\pi,\delta)\ [\pi,\delta]. \tag{29}$$

0.3.3. *Kazhdan-Lusztig conjecture.* Recall in [7, Section 6.3] we have defined a pairing

$$\langle\, \cdot\, , \cdot\, \rangle : \mathsf{K}\Pi_{\mathrm{pure},\lambda}(G/F) \times \mathsf{KPer}_{H_\lambda}(V_\lambda) \to \mathbb{Z}$$

such that for any $(\phi,\rho),(\phi',\rho') \in \Xi_\lambda({}^LG)$

$$\langle \pi(\phi,\rho), \mathcal{P}(\phi',\rho')\rangle = (-1)^{\dim C_\phi} e(\phi,\rho)\delta_{(\phi,\rho),(\phi',\rho')}$$

where $e(\phi,\rho)$ is the Kottwitz sign of G_δ determined by (ϕ,ρ). Kazhdan-Lusztig conjecture predicts that

$$\langle M(\phi,\rho), \mathcal{P}(\phi',\rho')\rangle = e(\phi,\rho)\delta_{(\phi,\rho),(\phi',\rho')}$$

for any $(\phi,\rho),(\phi',\rho') \in \Xi_\lambda({}^LG)$. We verify the Kazhdan-Lusztig conjecture in our examples. This is done by comparing the multiplicity matrix m_{rep} from Section 0.1.3 with the normalised geometric multiplicity matrix m'_{geo} from Section 0.2.3:

$${}^t m_{\mathrm{rep}} = m'_{\mathrm{geo}}.$$

As a consequence, we can verify [7, Conjecture 3] in our examples following the argument below. Let $\mathsf{K}_{\mathbb{C}}\Pi_{\mathrm{pure},\lambda}(G/F)^{\mathrm{st}}$ be the subspace of strongly stable virtual representations in $\mathsf{K}_{\mathbb{C}}\Pi_{\mathrm{pure},\lambda}(G/F) := \mathsf{K}\Pi_{\mathrm{pure},\lambda}(G/F) \otimes_{\mathbb{Z}} \mathbb{C}$. It has a natural basis

$$\eta_\phi := \sum_{\rho:(\phi,\rho)\in\Xi_\lambda({}^LG)} \dim(\rho) e(\phi,\rho) M(\phi,\rho)$$

parametrized by $\phi \in P_\lambda({}^LG)/H_\lambda$. After identifying $\mathsf{K}_{\mathbb{C}}\Pi_{\mathrm{pure},\lambda}(G/F)$ with

$$\mathsf{K}_{\mathbb{C}}\mathsf{Per}_{H_\lambda}(V_\lambda)^* = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{K}\mathsf{Per}_{H_\lambda}(V_\lambda), \mathbb{C}),$$

through the pairing above, we would like to characterize $\mathsf{K}_{\mathbb{C}}\Pi_{\mathrm{pure},\lambda}(G/F)^{\mathrm{st}}$ in $\mathsf{K}_{\mathbb{C}}\mathsf{Per}_{H_\lambda}(V_\lambda)^*$. By the Kazhdan-Lusztig conjecture,

$$\langle \eta_\phi, \mathcal{P}\rangle = \chi^{\mathrm{loc}}_{C_\phi}(\mathcal{P}) := \sum_{\rho:(\phi,\rho)\in\Xi_\lambda({}^LG)} (-1)^{\dim C_\phi} m_{\mathrm{geo}}(\mathcal{S}(C_\phi, \mathcal{L}_\rho), \mathcal{P}),$$

for any $\mathcal{P} \in \mathsf{K}\mathsf{Per}_{H_\lambda}(V_\lambda)$. Therefore, $\mathsf{K}_{\mathbb{C}}\Pi_{\mathrm{pure},\lambda}(G/F)^{st}$ is spanned by $\chi^{\mathrm{loc}}_{C_\phi}(\cdot)$ for $\phi \in P_\lambda({}^LG)/H_\lambda$ in $\mathsf{K}_{\mathbb{C}}\mathsf{Per}_{H_\lambda}(V_\lambda)^*$. On the other hand, by Ginzburg, Kashiwara and Dubson [6] [12], we know that for any $\phi \in P_\lambda({}^LG)$ and $\mathcal{P} \in \mathsf{K}\mathsf{Per}_{H_\lambda}(V_\lambda)$,

$$\chi^{mic}_{C_\phi}(\mathcal{P}) := \mathrm{rank}\, \mathsf{Ev}_{C_\phi}(\mathcal{P}) = \sum_{\phi' \in P_\lambda({}^LG)/H_\lambda} c(C_\phi, C_{\phi'}) \chi^{loc}_{C_{\phi'}}(\mathcal{P}),$$

where $c(C_\phi, C_{\phi'})$ satisfies the following properties: $c(C_\phi, C_\phi) = (-1)^{\dim C_\phi}$; and $c(C_\phi, C_{\phi'}) \neq 0$ only if $\bar{C}_{\phi'} \supseteq C_\phi$. The coefficients $c(C_\phi, C_{\phi'})$ are related to the local Euler obstructions defined by MacPherson. In particular, it measures the singularity of the closure of $C_{\phi'}$ at its boundary stratum C_ϕ. As a consequence, we see the set of $\chi^{mic}_{C_\phi}(\cdot)$ for $\phi \in P_\lambda({}^LG)/H_\lambda$ forms another basis for $\mathsf{K}\Pi_{\mathrm{pure},\lambda}(G/F)^{st}_{\mathbb{C}}$. Finally, it is easy to see that for any $\phi \in P_\lambda({}^LG)$ and $\mathcal{P} \in \mathsf{K}\mathsf{Per}_{H_\lambda}(V_\lambda)$

$$\langle \eta^{\mathsf{NEv}}_{C_\phi}, \mathcal{P}\rangle = (-1)^{\dim C_\phi} \chi^{mic}_{C_\phi}(\mathcal{P}).$$

So the set of $\eta^{\mathsf{NEv}}_{C_\phi}$ for $\phi \in P_\lambda({}^LG)/H_\lambda$ also forms a basis for $\mathsf{K}\Pi_{\mathrm{pure},\lambda}(G/F)^{st}_{\mathbb{C}}$. This proves [7, Conjecture 3].

0.3.4. *Aubert duality and Fourier transform.* In order to compare Aubert duality with the Fourier transform, we equip V with the symmetric bilinear form $(x, y) \mapsto -(\, x \mid {}^t y\,)$, where ^{t} refers to transposition in j_λ, and we use this to define an isomorphism $V \to V^*$. We use the notation $\hat{C} := {}^tC^*$. Let $\vartheta : H \to H$ be the isomorphism of algebraic groups given by $\vartheta(h) = {}^th^{-1}$, in which ^{t} refers to transposition in J_λ. Then $V \to V^*$ is equivariant for the usual action of H on V and the twist by ϑ of the usual action of H on V^*. Now, *equivariant* pullback defines an equivalence of categories $\mathsf{Per}_H(V^*) \to \mathsf{Per}_H(V)$. When pre-composed with the Fourier transform $\mathsf{Ft} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(V^*)$, this defines a functor denoted by ${}^\wedge : \mathsf{Per}_H(V) \to \mathsf{Per}_H(V)$. Our examples show

$$\mathcal{P}(\hat{\pi}, \delta) = \widehat{\mathcal{P}(\pi, \delta)}. \tag{30}$$

The local system $\mathcal{T}$ on $T^*_H(V^*)_{\mathrm{reg}}$ appearing in Sections 0.2.6 and 0.2.8 admits an interesting description on a subbundle of $T^*_H(V^*)_{\mathrm{reg}}$. Suppose C is of Arthur type, so $C = C_\psi$ for an Arthur parameter ψ with infinitesimal parameter λ. Since $\mathcal{T}_{C_\psi}$ is a rank-1 equivariant local system on $T^*_{C_\psi}(V)_{\mathrm{reg}}$, it defines a character of A_ψ, denoted here by $\mathrm{trace}\, \mathcal{T}_\psi$. In this paper we see in our examples that

$$\mathrm{trace}\, \mathcal{T}_\psi = \chi_\psi, \tag{31}$$

where χ_ψ is the character of A_ψ that appeared in Section 0.1.5.

Using the equivalence $\mathsf{Per}_H(V^*) \to \mathsf{Per}_H(V)$ described above, (21) may be re-written in form

$$ {}^p\mathsf{NEv}\,\mathcal{P} = {}^p\mathsf{Ev}\,\widehat{\mathcal{P}}, \tag{32}$$

Taking traces, and recalling ${}^p\mathsf{NEv} = \mathcal{T}^\vee \otimes {}^p\mathsf{Ev}$, this implies

$$\mathrm{trace}_a\left(\mathsf{NEv}_{\hat{C}}\,\widehat{\mathcal{P}}\right) = \mathrm{trace}_s\,\mathcal{T}_C\ \ \mathrm{trace}_a\left(\mathsf{NEv}_C\,\mathcal{P}\right)$$

for every $a \in A_C^{\mathrm{mic}}$. Taking $\mathcal{P} = \mathcal{P}(\pi)$ and $C = C_\psi$ and using (30) and (31), we recover (10).

0.3.5. *ABV-packets that are not pure Arthur packets.* While all pure Arthur packets are ABV-packets in these examples, it is not true that all ABV-packets are pure Arthur packets. In Sections 4.3.6 and 6.3.5 we discuss examples of ABV-packets that are not pure Arthur packets and yet enjoy many of the properties we expect from Arthur packets.

0.4. **Endoscopy and equivariant restriction of perverse sheaves.** One of the ingredients in the proof of [7, Conjecture 2] in [8] for unipotent representations of odd orthogonal groups, is the following theorem. Let G' be an endoscopic group for G though which $\lambda : W_F \to {}^LG$ factors, thus defining $\lambda' : W_F \to {}^LG'$. Set $V' = V_{\lambda'}$. Let C' be an H'-orbit in V'; pick $(x', \xi') \in T^*_{C'}(V')_{\mathrm{reg}}$ and let C be the H-orbit in V of the image of x' under $V' \hookrightarrow V$. Suppose that the conormal map $T^*_{C'}(V') \to T^*_C(V)$ restricts to $T^*_{C'}(V')_{\mathrm{reg}} \to T^*_C(V)_{\mathrm{reg}}$. Let $(x, \xi) \in T^*_C(V)_{\mathrm{reg}}$ be its image of $(x', \xi') \in T^*_{C'}(V')_{\mathrm{reg}}$ under that map. Then, for every $\mathcal{P} \in \mathsf{Per}_H(V)$,

$$(-1)^{\dim C'}\ \ \mathrm{trace}_{a'_s}\left(\mathsf{NEv}'\,\mathcal{P}|_{V'}\right)_{(x',\xi')} = (-1)^{\dim C}\ \ \mathrm{trace}_{a_s}\left(\mathsf{NEv}\,\mathcal{P}\right)_{(x,\xi)}, \tag{33}$$

where a_s is the image of s under $Z_{\widehat{G}}(x, \xi) \to A_{(x,\xi)}$ and a'_s is the image of s under $Z_{\widehat{G}'}(x', \xi') \to A_{(x',\xi')}$.

In the examples in this paper, we calculate both sides of (34), independently, in order to illustrate how the functor of vanishing cycles Ev interacts with the equivariant restriction functor $D_H(V) \to D_{H'}(V')$. As explained in [8], it is (34) that allows us to conclude that $\eta^{\mathsf{NEv}}_{\phi,s}$ is the endoscopic transfer of a strongly stable virtual representation on G'.

Although we don't show the calculations here, the same arguments used to prove (33) also show

$$(-1)^{e_{C'}-\dim C'}\ \ \mathrm{trace}_{a'_s}\left(\mathsf{Ev}'\,\mathcal{P}|_{V'}\right)_{(x',\xi')} = (-1)^{e_C-\dim C}\ \ \mathrm{trace}_{a_s}\left(\mathsf{Ev}\,\mathcal{P}\right)_{(x,\xi)}, \tag{34}$$

under the same hypotheses.

0.4.1. *Endoscopic Vogan varieties.* After recalling the endoscopic groups G' and the infinitesimal parameters $\lambda' : W_F \to {}^LG'$ such that $\lambda = \epsilon \circ \lambda'$ from Section 0.1.6, we describe $V' := V_{\lambda'}$ and its stratification into orbits under the action by $H' := Z_{\widehat{G}'}(\lambda')$. In all cases, $G' = G^{(2)} \times G^{(1)}$ so $\lambda' = (\lambda^{(2)}, \lambda^{(1)})$. Except for Section 1, we have arranged the sequence of examples so that by the time we get to $\lambda' : W_F \to {}^LG'$, both $\lambda^{(1)} : W_F \to {}^LG^{(1)}$ and $\lambda^{(2)} : W_F \to {}^LG^{(2)}$ have already been studied. Since $H' = H^{(2)} \times H^{(1)}$ and $V' = V^{(2)} \times V^{(1)}$, we use the equivalence

$$\mathsf{Per}_{H^{(2)}}(V^{(2)}) \times \mathsf{Per}_{H^{(1)}}(V^{(1)}) \xrightarrow{\boxtimes} \mathsf{Per}_{H'}(V')$$

to answer all questions about $\mathsf{Per}_{H'}(V')$ using earlier work.

The H'-invariant function $(\,\cdot\mid\cdot\,) : T^*(V') \to \mathbb{A}^1$ is simply the sum of the functions $T^*(V^{(1)}) \to \mathbb{A}^1$ and $T^*(V^{(2)}) \to \mathbb{A}^1$ while $[\,\cdot\,,\,\cdot\,] : T^*(V') \to \mathfrak{h}'$ is likewise built from the functions $T^*(V^{(1)}) \to \mathfrak{h}^{(1)}$ and $T^*(V^{(2)}) \to \mathfrak{h}_2$. Consequently, the conormal bundle is

$$T^*_{H'}(V') = T^*_{H^{(2)}}(V^{(2)}) \times T^*_{H^{(1)}}(V^{(2)}),$$

so $\mathsf{Per}_{H'}(T^*_{H'}(V')_{\mathrm{reg}})$ can be completely described using earlier work.

0.4.2. *Vanishing cycles.* It follows from the Thom-Sebastiani Theorem [17] that

$$\mathsf{Ev}'\left(\mathcal{IC}(C^{(2)},\mathcal{L}^{(2)}) \boxtimes \mathcal{IC}(C^{(1)},\mathcal{L}^{(1)})\right) = \left(\mathsf{Ev}\,\mathcal{IC}(C^{(2)},\mathcal{L}^{(2)})\right) \boxtimes \left(\mathsf{Ev}\,\mathcal{IC}(C^{(1)},\mathcal{L}^{(1)})\right).$$

Thus, the functor

$$\mathsf{Ev}' : \mathsf{Per}_{H'}(V') \to \mathsf{Per}_{H'}(T^*_{H'}(V')_{\mathrm{reg}})$$

may also be deduced from earlier work.

0.4.3. *Restriction.* The equivariant restriction functor

$$\begin{array}{ccc} \mathsf{D}_H(V) & \longrightarrow & \mathsf{D}_{H'}(V') \\ \mathcal{F} & \mapsto & \mathcal{F}|_{V'} \end{array} \tag{35}$$

does not take perverse sheaves to perverse sheaves. Since we wish to illustrate (34), we compute (35) in each example, after passing to Grothendieck groups.

0.4.4. *Restriction and vanishing cycles.* We have now assembled all the pieces needed to illustrate (34). We begin by identifying all $(x',\xi') \in T^*_{H'}(V')_{\mathrm{reg}}$ such that the image of (x',ξ') in $T^*_H(V)$ is regular. This gives us an opportunity to revisit the question of finding all Arthur parameters $\psi : L_F \times \mathrm{SL}(2) \to {}^LG$ with infinitesimal parameter λ that factor through $\epsilon : {}^LG' \to {}^LG$ to define Arthur parameters $\psi : L_F \times \mathrm{SL}(2) \to {}^LG$ with infinitesimal parameter λ'. Finally, for such (x',ξ') we pick a simple perverse sheaf $\mathcal{P} \in \mathsf{Per}_H(V)$ and compute both sides of (34), where s is determined by the elliptic endoscopic group G'.

1. SL(2) 4-packet of quadratic unipotent representations

Set $G = \mathrm{SL}(2)$ over F; so $\widehat{G} = \mathrm{PGL}(2,\mathbb{C})$ and ${}^LG = \mathrm{PGL}(2,\mathbb{C}) \times W_F$. Suppose q is odd.

The function $H^1(F,G) \to H^1(F,G_{\mathrm{ad}})$ is injective but not surjective; indeed, $H^1(F,G)$ is trivial but $H^1(F,G^*_{\mathrm{ad}}) \cong \mu_2$. In other words, SL(2) has no pure rational forms but it does have an inner rational form.

Let $\varpi \in F$ be a uniformizer and let $u \in F$ be a non-square unit integer. Let E/F be the biquadratic extension $E = F(\sqrt{\varpi},\sqrt{u})$. Then $\mathrm{Gal}(E/F) = \{1,\sigma,\tau,\sigma\tau\}$ where $\sigma(\sqrt{u}) = -\sqrt{u}$ and $\tau(\sqrt{\varpi}) = -\sqrt{\varpi}$. Define $\varrho : \mathrm{Gal}(E/F) \to \mathrm{PGL}(2)$ by

$$\sigma \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \tau \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\lambda : W_F \to {}^LG$ be the infinitesimal parameter defined by the composition $L_F \to W_F \to \Gamma_F \to \mathrm{Gal}(E/F)$ followed by $\varrho : \mathrm{Gal}(E/F) \to \widehat{G}$; thus,

$$\lambda(w) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} w \in {}^LG, \qquad \text{if } w|_E = \sigma,$$

and

$$\lambda(w) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w \in {}^LG, \qquad \text{if } w|_E = \tau.$$

1.1. **Arthur packets.**

1.1.1. *Parameters.* There is only one Langlands parameter with infinitesimal parameter λ chosen above: $\phi(w,x)=\lambda(w)$. This Langlands parameter is of Arthur type: $\psi(w,x,y)=\lambda(w)$.

1.1.2. *L-packets.* With ϕ as above, we have

$$Z_{\widehat{G}}(\phi)=\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\-1&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}.$$

Let $A_\phi\cong\mu_2\times\mu_2$ be the isomorphism determined by

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\mapsto(-1,+1)\qquad\text{and}\qquad\begin{pmatrix}1&0\\0&-1\end{pmatrix}\mapsto(+1,-1).$$

Using this isomorphism, the characters of A_ϕ will be denoted by $(++)$, $(+-)$, $(-+)$ and $(--)$. The L-packet $\Pi_\phi(G(F))$ is the unique cardinality-4 L-packet for $\mathrm{SL}(2,F)$:

$$\Pi_\phi(G(F))=\{\pi(\phi,++),\pi(\phi,+-),\pi(\phi,-+),\pi(\phi,--)\}.$$

This L-packet, which is described in [20, Section 11], may be obtained by restricting a supercuspidal representation of $\mathrm{GL}(2,F)$ given in [11, Theorem 4.6] to $\mathrm{SL}(2,F)$. Alternately, these depth-zero supercuspidal representations are produced by compact induction from a maximal parahoric (there are two, up to $G(F)$-conjugation), from (the inflation of) the two cuspidal irreducible representations appearing in the only non-singleton Deligne-Lusztig representation of $\mathrm{SL}(2,\mathbb{F}_q)$. The characters of these representations are described in [2, §15].

Since G has no pure inner forms, the pure packet for the Langlands parameter ϕ is an L-packet:

$$\Pi_{\mathrm{pure},\phi}(G/F)=\Pi_\phi(G(F)).$$

1.1.3. *Arthur packets.* The L-packet $\Pi_\phi(G(F))$ is an Arthur packet:

$$\Pi_{\mathrm{pure},\psi}(G/F)=\Pi_{\mathrm{pure},\phi}(G/F).$$

1.1.4. *Aubert duality.* Aubert involution fixes all the representations in this example.

1.1.5. *Stable distributions and endoscopy.* Since ψ is trivial on $\mathrm{SL}(2)$ in its domain, it follows that $s_\psi=1$, so the stable invariant distribution (12) attached to ψ is

$$\Theta_\psi=\operatorname{trace}\pi(\phi,++)+\operatorname{trace}\pi(\phi,+-)+\operatorname{trace}\pi(\phi,-+)+\operatorname{trace}\pi(\phi,--).$$

For any $s\in Z_{\widehat{G}}(\psi)$ (the 4-group $Z_{\widehat{G}}(\phi)$ appearing in Section 1.1.2) the coefficients of $\Theta_{\psi,s}$ appearing in (13) are simply

$$\langle s,\pi(\phi,\pm\pm)\rangle_\psi=(\pm\pm)(s).\tag{36}$$

Besides G itself, three endoscopic groups are relevant to ψ: the unramified torus $\mathrm{U}(1)$ split over $F(\sqrt{u})$, and the two ramified tori split over $F(\sqrt{\varpi})$, and $F(\sqrt{u\varpi})$. More precisely, in the case of the unramified torus, take $s\in\widehat{G}$ to be

$$s=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

and set

$$n=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Note that

$$s = \psi^\circ(w), \qquad \text{if} \quad w|_E = \tau$$

and

$$n = \psi^\circ(w), \qquad \text{if} \quad w|_E = \sigma.$$

Let G' be the endoscopic group U(1) split over $F(\sqrt{u})$ with: $\widehat{G'} = Z_{\widehat{G}}(s)^0$; action of W_F on $\widehat{G'}$ determined by $\pi_0(Z_{\widehat{G}}(s)) \cong \mathrm{Gal}(F(\sqrt{u})/F)$; and $\varepsilon : {}^LG' \to {}^LG$ given by $\widehat{G'} = Z_{\widehat{G}}(s)^0 \subset \widehat{G}$ and

$$\varepsilon(1 \rtimes w) := nw, \qquad \text{if} \quad w|_E = \sigma.$$

Then the Arthur parameter $\psi : L_F \times \mathrm{SL}(2) \to {}^LG$ factors through $\varepsilon : {}^LG' \to {}^LG$ to define $\psi' : L_F \times \mathrm{SL}(2) \to {}^LG'$, so

$$\psi'(w) = s \rtimes w \in {}^LG', \qquad \text{if} \quad w|_E = \tau.$$

The representation of $G'(F)$ with Arthur parameter ψ' is the quadratic character attached to the extension $F(\sqrt{u})/F$ by class field theory. Then the endoscopic transfer of the quadratic character from $G'(F)$ to $G(F)$ is

$$\Theta_{\psi,s} \quad = \quad \mathrm{trace}\,\pi(\phi,++) - \mathrm{trace}\,\pi(\phi,+-) + \mathrm{trace}\,\pi(\phi,-+) - \mathrm{trace}\,\pi(\phi,--).$$

The set-up is similar for the ramified tori, as we now explain. Take

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{respectively,} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and, in the same order, set

$$n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{respectively,} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$s = \psi^\circ(w), \qquad \text{if} \quad w|_E = \sigma, \quad \text{respectively,} \;\; w|_E = \sigma\tau,$$

and

$$n = \psi^\circ(w), \qquad \text{if} \quad w|_E = \sigma\tau, \quad \text{respectively,} \;\; w|_E = \tau.$$

Let G' be the endoscopic group U(1) split over $F(\sqrt{\varpi})$, respectively, $F(\sqrt{u\varpi})$ with: $\widehat{G'} = Z_{\widehat{G}}(s)^0$; action of W_F on $\widehat{G'}$ determined by

$$\pi_0(Z_{\widehat{G}}(s)) \cong \mathrm{Gal}(F(\sqrt{\varpi})/F), \quad \text{respectively,} \;\; \pi_0(Z_{\widehat{G}}(s)) \cong \mathrm{Gal}(F(\sqrt{u\varpi})/F);$$

and $\varepsilon : {}^LG' \to {}^LG$ given by $\widehat{G'} = Z_{\widehat{G}}(s)^0 \subset \widehat{G}$ and

$$\varepsilon(1 \rtimes w) := nw, \qquad \text{if} \quad w|_E = \sigma\tau, \quad \text{respectively,} \;\; w|_E = \tau.$$

Then the Arthur parameter $\psi : L_F \times \mathrm{SL}(2) \to {}^LG$ factors through $\varepsilon : {}^LG' \to {}^LG$ to define $\psi' : L_F \times \mathrm{SL}(2) \to {}^LG'$, so

$$\psi'(w) = s \rtimes w \in {}^LG', \qquad \text{if} \quad w|_E = \sigma, \quad \text{respectively,} \;\; w|_E = \sigma\tau.$$

The representation of $G'(F)$ with Arthur parameter ψ' is the quadratic character attached to the extension $F(\sqrt{\varpi})/F$, respectively, $F(\sqrt{u\varpi})/F$, by class field theory. Then the endoscopic transfer of the quadratic character from $G'(F)$ to $G(F)$ is $\Theta_{\psi,s}$ which, in order, is

$$\begin{aligned} \Theta_{\psi,s} \quad &= \quad \mathrm{trace}\,\pi(\phi,++) + \mathrm{trace}\,\pi(\phi,+-) - \mathrm{trace}\,\pi(\phi,-+) - \mathrm{trace}\,\pi(\phi,--), \qquad \text{respectively,} \\ \Theta_{\psi,s} \quad &= \quad \mathrm{trace}\,\pi(\phi,++) - \mathrm{trace}\,\pi(\phi,+-) - \mathrm{trace}\,\pi(\phi,-+) + \mathrm{trace}\,\pi(\phi,--). \end{aligned} \tag{37}$$

Together with the stable distribution Θ_ψ, these three $\Theta_{\psi,s}$ form a basis for the vector space spanned by the characters of representations with infinitesimal parameter λ. These four distributions are expressed in terms of the Fourier transform of regular semisimple orbital integrals, and their endoscopic transfer, in [9, §6.2].

1.1.6. *Jacquet-Langlands.* The L-packet that this example treats also appears in [5, §4, page 215], alongside the L-packet for the inner form corresponding to a non-trivial cocycle in $Z^1(F, G_{\mathrm{ad}})$, which determines the compact form of G, mentioned at the beginning of this section and now denoted by G_σ. The same Langlands parameter ϕ as above, when viewed as a Langlands parameter for G_σ, produces a singleton L-packet. In this case $\mathcal{S}_{\psi,\mathrm{sc}} = Z_{\widehat{G}_{\mathrm{sc}}}(\psi)$, which is the subgroup of $\widehat{G}_{\mathrm{sc}} = \mathrm{SL}(2)$ isomorphic to Q_8 given by

$$\mathcal{S}_{\psi,\mathrm{sc}} = \left\{ \begin{array}{cccc} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, & \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, & \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{array} \right\}.$$

The compact form G_σ of $G = \mathrm{SL}(2)$ carries exactly one admissible representation with infinitesimal parameter λ, and it corresponds to the unique irreducible 2-dimensional representation of this group. We denote this representation by $\pi(\phi, 2)$. Although the theory presented in [7] does include inner rational forms that are not pure, in Section 1.2.7 we will show how to adapt the geometric picture so that it does include $\pi(\phi, 2)$.

1.2. **Vanishing cycles of perverse sheaves.**

1.2.1. *Vogan variety and orbit duality.* Recall the groups H_λ, J_λ and K_λ from [7, Section 3]. In the example at hand, these are given by

$$H_\lambda = J_\lambda = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \cong \mu_2 \times \mu_2$$

and $K_\lambda = N_{\widehat{G}^*}(\widehat{T})$. In particular, $G_\lambda = 1$ and $\lambda_{\mathrm{nr}} : W_F \to {}^L G$ is trivial so $V_{\lambda_{\mathrm{nr}}} = 0$ and $H_{\lambda_{\mathrm{nr}}} = 1$.

1.2.2. *Equivariant perverse sheaves.* With reference to [7, Theorem 3.1.1] we have

$$\begin{array}{ccccc} \mathsf{Rep}(A_\lambda) & \xrightarrow{\text{equiv.}} & \mathsf{Per}_{H_\lambda}(V_\lambda) & \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} & \mathsf{Per}_{H_{\lambda_{\mathrm{nr}}}}(0) \\ \| & & \| & & \| \\ \mathsf{Rep}(\mu_2 \times \mu_2) & & \mathsf{Per}_{\mu_2 \times \mu_2}(0) & & \mathsf{Per}_1(0). \end{array}$$

In particular, there are four simple objects in $\mathsf{Per}_{H_\lambda}(V_\lambda)$ corresponding to the four simple H_λ-equiviariant local systems on $V_\lambda = \{0\}$, or equivalently, to the four characters of A_λ:

$$\mathsf{Per}_H(V)^{\mathrm{simple}}_{/\mathrm{iso}} = \{(++)_V, (+-)_V, (-+)_V, (--)_V\}.$$

1.2.3. *Vanishing cycles of perverse sheaves.* We wish to describe the functor

$$\mathsf{Ev} : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}}).$$

We have already seen that $\mathsf{Per}_{H_\lambda}(V_\lambda) = \mathsf{Rep}(A_\lambda)$. In this case we have $T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}} = \{(0,0)\}$, so $\mathsf{Per}_{H_\lambda}(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}}) = \mathsf{Rep}(A_\lambda)$. With these equivalences,

$$\mathsf{Ev} : \mathsf{Rep}(A_\lambda) \to \mathsf{Rep}(A_\lambda)$$

is the identity functor:

$$\mathsf{Ev}_\psi(\pm\pm)_V(s) = (\pm\pm)(s) \tag{38}$$

for every $s \in Z_{\widehat{G}}(\psi)$.

1.2.4. *Normalization of Ev and the twisting local system.* Since Ev is trivial in this example, so is $\mathcal{T}$ and NEv; accordingly, the material of Section 0.2.7 is trivial in this example.

1.2.5. *Fourier transform and vanishing cycles.* Since Ev, NEv and Ft are trivial in this example, the material of Section 0.2.8 is trivial.

1.2.6. *Arthur sheaves.* Since $V_\lambda = \{0\}$ is a single stratum, there is only one stable perverse sheaf to consider:

$$\mathcal{A}_{C_0} = (++)_V \oplus (+-)_V \oplus (-+)_V \oplus (--)_V.$$

Of course, this is just the regular representation of A_λ.

1.2.7. *Jacquet-Langlands.* We now show how to extend the geometric picture to include the admissible representation $\pi(\phi, 2)$ of the inner rational form G_σ of G.

Replace the group action $H_\lambda \times V_\lambda \to V_\lambda$ with the group action $H_{\lambda,\mathrm{sc}} \times V_\lambda \to V_\lambda$, where

$$H_{\lambda,\mathrm{sc}} := Z_{\widehat{G}_{\mathrm{sc}}}(\lambda),$$

and where $H_{\lambda,\mathrm{sc}}$ acts on V_λ through $H_{\lambda,\mathrm{sc}} \to H_\lambda$ induced by the universal cover $\widehat{G}_{\mathrm{sc}} \to \widehat{G}$. The analysis of [7, Section Fix] shows that

$$\mathsf{Per}_{H_{\lambda,\mathrm{sc}}}(V_\lambda) \equiv \mathsf{Rep}(A_{\lambda,\mathrm{sc}}),$$

where $A_{\lambda,\mathrm{sc}} := \pi_0(H_{\lambda,\mathrm{sc}})$. Of course, $A_{\lambda,\mathrm{sc}}$ is just the group $\mathcal{S}_{\psi,\mathrm{sc}}$ appearing above. Now $A_{\lambda,\mathrm{sc}}$ has five irreducible representations up to equivalence: four one-dimensional representations obtained by pullback from the four characters of A_λ we have already seen, and one two dimensional representation, denoted by E. Thus, the category $\mathsf{Rep}(A_{\lambda,\mathrm{sc}})$ has exactly five simple objects up to isomorphism, and thence $\mathsf{Per}_{H_{\lambda,\mathrm{sc}}}(V_\lambda)$ has exactly five simple objects up to isomorphism:

$$\mathsf{Per}^{\mathrm{simple}}_{H_{\lambda,\mathrm{sc}}}(V_\lambda)_{/\mathrm{iso}} = \{E_V, (++)_V, (+-)_V, (-+)_V, (--)_V\}.$$

The rest of the story now carries through. For instance, the diagram of functors from Section 1.2.2 becomes the following diagram:

$$\begin{array}{ccccc}
\mathsf{Rep}(A_{\lambda,\mathrm{sc}}) & \xrightarrow{\text{equiv.}} & \mathsf{Per}_{H_{\lambda,\mathrm{sc}}}(V_\lambda) & \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} & \mathsf{Per}_{H^0_{\lambda,\mathrm{sc}}}(V_\lambda) \\
\| & & \| & & \| \\
\mathsf{Rep}(Q_8) & & \mathsf{Per}_{Q_8}(0) & & \mathsf{Per}_1(0).
\end{array}$$

Also, the functor vanishing cycles, Ev, is again the identity functor $\mathsf{Rep}(A_{\lambda,\mathrm{sc}}) \to \mathsf{Rep}(A_{\lambda,\mathrm{sc}})$, and the Arthur sheaf is again just the regular representation of $A_{\psi,\mathrm{sc}}$. Thus, simply replacing category $\mathsf{Per}_{H_\lambda}(V_\lambda)$ with $\mathsf{Per}_{H_{\lambda,\mathrm{sc}}}(V_\lambda)$ extends the theory from pure inner twists of G to inner twists of G, allowing us to see the Jacquet-Langlands correspondence in the geometric perspective of [7].

1.3. **Adams-Barbasch-Vogan packets.**

1.3.1. *Admissible representations versus equivariant perverse sheaves.* The following table displays Vogan's bijection between $\mathsf{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}}$ and $\Pi_{\text{pure},\lambda}(G/F)$, as discussed in Section 0.3.1.

$\mathsf{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}}$	$\Pi_{\text{pure},\lambda}(G/F)$
$(++)_V$	$\pi(\phi,++)$
$(+-)_V$	$\pi(\phi,+-)$
$(-+)_V$	$\pi(\phi,-+)$
$(--)_V$	$\pi(\phi,--)$

1.3.2. *ABV-packets.* Using the bijection from Section 1.3.1 and the trivial functor of Ev from Section 1.2.3, it follows directly from definition (25) that

$$\Pi_\psi(G(F)) = \Pi^{\text{ABV}}_{\text{pure},\phi_\psi}(G/F).$$

With reference to (29) and (38), in this example we find

$$\begin{aligned}
\eta^{\mathsf{NEv}}_{\psi,s} &= \operatorname{trace}\pi(\phi,++) - \operatorname{trace}\pi(\phi,+-) + \operatorname{trace}\pi(\phi,-+) - \operatorname{trace}\pi(\phi,--),\\
\eta^{\mathsf{NEv}}_{\psi,s} &= \operatorname{trace}\pi(\phi,++) + \operatorname{trace}\pi(\phi,+-) - \operatorname{trace}\pi(\phi,-+) - \operatorname{trace}\pi(\phi,--), \qquad \text{and then}\\
\eta^{\mathsf{NEv}}_{\psi,s} &= \operatorname{trace}\pi(\phi,++) - \operatorname{trace}\pi(\phi,+-) - \operatorname{trace}\pi(\phi,-+) + \operatorname{trace}\pi(\phi,--).
\end{aligned}$$

Comparing $\eta^{\mathsf{NEv}}_{\psi,s}$ above with $\eta_{\psi,s}$ as calculated in Section 1.1.5 in (36) and (37), we see that

$$\eta^{\mathsf{NEv}}_{\psi,s} = \eta_{\psi,s},$$

in all four cases, thus confirming [7, Conjecture 2] in this example.

1.3.3. *Kazhdan-Lusztig conjecture.* The material of Section 0.3.3 is trivial in this example.

1.4. **Endoscopy and equivariant restriction of perverse sheaves.** In Section 1.1.5 we saw that the Arthur parameter ψ factors though three elliptic endoscopic groups, G'. For each of these G', the infinitesimal parameter $\lambda : W_F \to {}^LG$ factors through $\varepsilon : {}^LG' \to {}^LG$ to define $\lambda' : W_F \to {}^LG'$.

1.4.1. *Endoscopic Vogan variety.* For each G' above, $H' := Z_{\widehat{G'}}(\lambda')$ is the subgroup of H generated by s in H'; see Section 1.1.5 for s. Thus, $\mathsf{Per}_{H'}(V') \equiv \mathsf{Rep}(H')$ has two simple objects, now denoted by $(+)_{V'}$ and $(-)_{V'}$. Now, Vogan's bijection for $\lambda' : W_F \to {}^LG'$ is given by the following table.

$\mathsf{Per}_{H'}(V')^{\text{simple}}_{/\text{iso}}$	$\Pi_{\text{pure},\lambda'}(G'/F)$
$(+)_{V'}$	$\pi(\phi',+)$
$(-)_{V'}$	$\pi(\phi',+)$

Then $\pi(\phi',+) = \pi(\phi',-)$ is the quadratic character of $G'(F) = N^{-1}_{E'/F}(1)$ determined by ϕ'.

1.4.2. *Vanishing cycles.* Arguing as in Section 1.2.3, we see that

$$\mathsf{NEv}' : \mathsf{Rep}(A_{\lambda'}) \to \mathsf{Rep}(A_{\lambda'})$$

is trivial.

1.4.3. *Restriction.* The restriction functor $\mathsf{Per}_H(V) \to \mathsf{Per}_{H'}(V')$ is just restriction $\mathsf{Rep}(H) \to \mathsf{Rep}(H')$ to the subgroup generated by s.

1.4.4. *Restriction and vanishing cycles.* We see (34) almost trivially: the left-hand side of (34) is

$$\mathrm{trace}_{a_s} \mathsf{NEv}_\psi(\pm\pm)_V = (\pm\pm)(s)$$

while the right-hand side of (34) is

$$(-1)^{\dim C - \dim C'} \ \mathrm{trace}_{a'_s} \left(\mathsf{Ev}'_{\psi'}(\pm\pm)_V|_{V'}\right) = (-1)^{0-0}(\pm\pm)(s).$$

Arguing as in [8], it follows from (34) that $\eta^{\mathsf{NEv}}_{\psi,s}$ is the Langlands-Shelstad lift of $\eta^{\mathsf{NEv}}_{\psi'}$. These lifts are found by considering each case in turn: in order, take $s \in \widehat{G}$ to be

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and then} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

in the same order, the quadratic extension E'/F is

$$E'/F = F(\sqrt{u})/F, \ F(\sqrt{\varpi})/F, \quad \text{and then} \quad F(\sqrt{u\varpi})/F.$$

2. SO(3) unipotent representations, regular infinitesimal parameter

Set $G = \mathrm{SO}(3)$ split over F, so $\widehat{G} = \mathrm{SL}(2,\mathbb{C})$ and ${}^LG = \mathrm{SL}(2,\mathbb{C}) \times W_F$. In this case,

$$H^1(F,G) = H^1(F,G_{\mathrm{ad}}) = H^1(F,\mathrm{Aut}(G)) \cong \mathbb{Z}/2\mathbb{Z},$$

so there are two isomorphism classes of rational forms of G, each pure. We will use the notation $G_0 = G$ for the split form and G_1 for the non-quasisplit form of SO(3) given by the quadratic form

$$\begin{pmatrix} -\varepsilon\varpi & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varpi \end{pmatrix}.$$

Let $\lambda : W_F \to \widehat{G}$ be the parameter defined by

$$\lambda(w) = \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}.$$

Even this simple example exhibits some interesting geometric phenomena, but the Arthur packets in this example are singletons, so there is no interesting endoscopy here. Nevertheless, this example will be important later when we consider other groups for which SO(3) is an endoscopic group.

2.1. **Arthur packets.**

2.1.1. *Parameters.* Up to $Z_{\widehat{G}}(\lambda)$-conjugacy, there are two Langlands parameters $\phi : L_F \to \widehat{G}$ with infinitesimal parameter λ; they are given by

$$\phi_0(w,x) = \lambda(w) = \nu_2(d_w) \qquad \text{and} \qquad \phi_1(w,x) = \nu_2(x),$$

where $\nu_2 : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C})$ is the identity function, thus an irreducible 2-dimensional representation of $\mathrm{SL}(2,\mathbb{C})$. So,

$$P_\lambda({}^LG)/Z_{\widehat{G}}(\lambda) = \{\phi_0, \phi_1\}.$$

Both ϕ_0 and ϕ_1 are of Arthur type: define

$$\psi_0(w,x,y) := \nu_2(y) \qquad \text{and} \qquad \psi_1(w,x,y) := \nu_2(x).$$

Then

$$Q_\lambda({}^LG)/Z_{\widehat{G}}(\lambda) = \{\psi_0, \psi_1\}.$$

Observer that ψ_1 is tempered but ψ_0 is not. Also observe that the Arthur parameters ψ_0 and ψ_1 are Aubert dual to each other.

2.1.2. *L-packets.* The component groups for the parameters $\phi \in P_\lambda({}^LG)$ are

$$A_{\phi_0} = \pi_0(Z_{\widehat{G}}(\phi_0)) = \pi_0(\widehat{T}) \cong 1 \qquad \text{and} \qquad A_{\phi_1} \cong \pi_0(Z_{\widehat{G}}(\phi_1)) = \pi_0(Z(\widehat{G})) \cong \mu_2.$$

Denoting the two characters of μ_2 by $+$ and $-$, the L-packets for these Langlands parameters are:

$$\begin{array}{rclrcl} \Pi_{\phi_0}(G_0(F)) & = & \{\pi(\phi_0)\}, & \Pi_{\phi_1}(G_0(F)) & = & \{\pi(\phi_1,+)\}, \\ \Pi_{\phi_0}(G_1(F)) & = & \emptyset, & \Pi_{\phi_1}(G_1(F)) & = & \{\pi(\phi_1,-)\}. \end{array}$$

Here we can view these representations as that of $\mathrm{GL}(2,F)$ (resp. multiplicative group of the quaternion algebra D) with trivial central character for $G(F) \cong \mathrm{GL}(2,F)/F^\times$ (resp. $G_1(F) \cong D^\times/F^\times$). Then $\pi(\phi_0)$ (resp. $\pi(\phi_1,+)$) is given by the trivial (resp. Steinberg) representation of $\mathrm{GL}(2,F)$ and $\pi(\phi_1,-)$ is given by the trivial representation of $D^\times$.

To see how characters ρ of A_ϕ determine pure inner forms of G, pullback ρ along $\pi_0(Z(\widehat{G})) \to \pi_0(Z_{\widehat{G}}(\phi))$ and then use the Kottwitz isomorphism: the trivial character of A_{ϕ_0} (resp. A_{ϕ_1}) determines the trivial character of $\pi_0(Z(\widehat{G}))$ and therefore the split pure inner form of G; the non-trivial character $-$ of A_{ϕ_1} determines the non-trivial character of $\pi_0(Z_{\widehat{G}})$ and therefore the non-trivial pure inner form of G. Therefore, the pure L-packets are:

$$\Pi_{\mathrm{pure},\phi_0}(G/F) \;=\; \{[\pi(\phi_0),0]\}, \qquad \Pi_{\mathrm{pure},\phi_1}(G/F) \;=\; \left\{ \begin{array}{c} [\pi(\phi_1,+),0] \\ [\pi(\phi_1,-),1] \end{array} \right\}.$$

2.1.3. *Multiplicities in standard modules.*

	$\pi(\phi_0)$	$\pi(\phi_1,+)$	$\pi(\phi_1,-)$
$M(\phi_0)$	1	1	0
$M(\phi_1,+)$	0	1	0
$M(\phi_1,-)$	0	0	1

2.1.4. *Arthur packets.* The component groups A_{ψ_0} and A_{ψ_1} are both $Z(\widehat{G})$. The Arthur packets for $\psi \in Q_\lambda({}^LG)$ are

$$\begin{array}{rclrcl} \Pi_{\psi_0}(G_0(F)) & = & \{\pi(\phi_0)\}, & \Pi_{\psi_1}(G_0(F)) & = & \{\pi(\phi_1,+)\}, \\ \Pi_{\psi_0}(G_1(F)) & = & \{\pi(\phi_1,-)\}, & \Pi_{\psi_1}(G_1(F)) & = & \{\pi(\phi_1,-)\}. \end{array}$$

so the pure Arthur packets are

$$\Pi_{\mathrm{pure},\psi_0}(G/F) \;=\; \left\{ \begin{array}{c} [\pi(\phi_0),0] \\ [\pi(\phi_1,-),1] \end{array} \right\}, \qquad \Pi_{\mathrm{pure},\psi_1}(G/F) \;=\; \left\{ \begin{array}{c} [\pi(\phi_1,+),0] \\ [\pi(\phi_1,-),1] \end{array} \right\}.$$

2.1.5. *Aubert duality.* Aubert duality for $G_0(F)$ and $G_1(F)$ is given by the following table.

π	$\hat{\pi}$
$\pi(\phi_0)$	$\pi(\phi_1,+)$
$\pi(\phi_1,-)$	$\pi(\phi_1,-)$

The twisting character χ_{ψ_0} of A_{ψ_0} is trivial; likewise, the twisting character χ_{ψ_1} of A_{ψ_1}.

2.1.6. *Stable distributions and endoscopy.* The characters $\langle\,\cdot\,,\pi\rangle_\psi$ appearing in the invariant distributions $\Theta^G_{\psi,s}$ (13) are given by the first two rows of the following table. The last row gives the analogous characters for $\Theta^{G_1}_{\psi,s}$.

π	$\langle\,\cdot\,,\pi\rangle_{\psi_0}$	$\langle\,\cdot\,,\pi,\rangle_{\psi_1}$
$\pi(\phi_0)$	$+$	0
$\pi(\phi_1,+)$	0	$+$
$\pi(\phi_1,-)$	$-$	$-$

Using the notation $s=\mathrm{diag}(s_1,s_1)\in A_\psi=Z(\widehat{G})$, we now have

$$\begin{array}{rclrcl}\Theta^G_{\psi_0,s} &=& \mathrm{trace}\,\pi(\phi_0), & \Theta^{G_1}_{\psi_0,s} &=& -s_1\,\mathrm{trace}\,\pi(\phi_1,-),\\ \Theta^G_{\psi_1,s} &=& \mathrm{trace}\,\pi(\phi_1,+), & \Theta^{G_1}_{\psi_1,s} &=& s_1\,\mathrm{trace}\,\pi(\phi_1,-).\end{array}$$

Therefore, in this example, the virtual representations $\eta_{\psi,s}$ (16) are:

$$\begin{array}{rcl}\eta_{\psi_0,s} &=& \pi(\phi_0)+s_1\pi(\phi_1,-),\\ \eta_{\psi_1,s} &=& \pi(\phi_1,+)-s_1\pi(\phi_1,-).\end{array}$$

Since $A_\psi=Z(\widehat{G})$, the only endoscopic groups relevant to these parameters is $G=G_0$.

2.2. **Vanishing cycles of perverse sheaves.**

2.2.1. *Vogan variety and orbit duality.* Since $\lambda: W_F\to {}^LG$ is unramified and $\lambda(\mathrm{Fr})$ is elliptic and G is split, we have $\lambda_{\mathrm{nr}}=\lambda$.

The Vogan variety for λ is

$$V_\lambda=\left\{\begin{pmatrix}0&y\\0&0\end{pmatrix}\in\widehat{\mathfrak{g}}\mid y\right\}\cong\mathbb{A}^1,$$

with $H_\lambda:=Z_{\widehat{G}}(\lambda)$-action

$$\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}\;:\;\begin{pmatrix}0&y\\0&0\end{pmatrix}\mapsto\begin{pmatrix}0&t^2y\\0&0\end{pmatrix}$$

so V_λ is stratified into H_λ-orbits

$$C_0:=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix}\right\}\qquad\text{and}\qquad C_y:=\left\{\begin{pmatrix}0&y\\0&0\end{pmatrix}\in\widehat{\mathfrak{g}}\;\mid\;y\neq0\right\}.$$

The dual Vogan variety V^*_λ is given by

$$V^*_\lambda=\left\{\begin{pmatrix}0&0\\y'&0\end{pmatrix}\in\widehat{\mathfrak{g}}\mid y'\right\}\cong\mathbb{A}^1,$$

with H_λ-action

$$\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}\;:\;\begin{pmatrix}0&0\\y'&0\end{pmatrix}\mapsto\begin{pmatrix}0&0\\t^{-2}y'&0\end{pmatrix},$$

so V^*_λ is stratified into H_λ-orbits

$$C^t_0:=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix}\right\}\qquad\text{and}\qquad C^t_y:=\left\{\begin{pmatrix}0&0\\y'&0\end{pmatrix}\in\widehat{\mathfrak{g}}\;\mid\;y'\neq0\right\}$$

The H_λ-invariant function $[\,\cdot\,,\,\cdot\,]:T^*(V_\lambda)\to\mathfrak{h}_\lambda$ is given by

$$\begin{pmatrix}0&y\\y'&0\end{pmatrix}\mapsto yy'\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

From this, dual orbits are easily found.

$$\begin{array}{ccc} C_y = \widehat{C}_0 & \dim = 1 & C_0^* = C_y^t \\ \uparrow & & \uparrow \\ C_0 = \widehat{C}_y & \dim = 0 & C_y^* = C_0^t \end{array}$$

2.2.2. *Equivariant perverse sheaves.* On the closed stratum C_0 there is one simple local system $\mathbb{1}_{C_0}$ and its perverse extension $\mathcal{IC}(\mathbb{1}_{C_0})$ is the rank-one skyscraper sheaf at C_0. The open stratum C_y carries two simple local systems: $\mathbb{1}_{C_y}$ and the non-trivial $\mathcal{E}_{C_y}$ corresponding, respectively, to the trivial and non-trivial characters of the equivariant fundamental group of C_y. Therefore, the irreducible shifted standard sheaves on V are:

$$\begin{array}{rcl} \mathcal{S}(\mathbb{1}_{C_0}) & = & j_{C_0!}\mathbb{1}_{C_0}[0], \\ \mathcal{S}(\mathbb{1}_{C_y}) & = & j_{C_y!}\mathbb{1}_{C_y}[1], \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathcal{S}(\mathcal{E}_{C_y}) & = & j_{C_y!}\mathcal{E}_{C_y}[1]. \end{array}$$

There are three simple objects in $\mathsf{Per}_{H_\lambda}(V_\lambda) = \mathsf{Per}_{\mathbb{G}_\mathrm{m}}(\mathbb{A}^1)$ up to isomorphism:

$$\mathsf{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}} = \left\{\mathcal{IC}(\mathbb{1}_{C_0}),\ \mathcal{IC}(\mathbb{1}_{C_y}),\ \mathcal{IC}(\mathcal{E}_{C_y})\right\}.$$

The perverse extension of $\mathbb{1}_{C_y}$ is the constant sheaf $\mathbb{1}_{V_\lambda}[1] = \mathcal{IC}(\mathbb{1}_{C_y})$ while the perverse extension $\mathcal{IC}(\mathcal{E}_{C_y})$ of $\mathcal{E}_{C_y}$ is the standard sheaf obtained by extension by zero from $\mathcal{E}_{C_y}[1]$.

$\mathcal{P}$	$\mathcal{P}\vert_{C_0}$	$\mathcal{P}\vert_{C_1}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathbb{1}_{C_0}[0]$	0
$\mathcal{IC}(\mathbb{1}_{C_y})$	$\mathbb{1}_{C_0}[1]$	$\mathbb{1}_{C_y}[1]$
$\mathcal{IC}(\mathcal{E}_{C_y})$	0	$\mathcal{E}_{C_y}[1]$

The first two row of this table are clear since $\overline{C_0}$ and $\overline{C_y}$ are smooth. To see the third row, let $\pi : V \to V$ be the proper double cover given by $y \mapsto y^2$ and note that

$$\pi_*(\mathbb{1}_V[1]) = \mathcal{IC}(\mathbb{1}_{C_y}) \oplus \mathcal{IC}(\mathcal{E}_{C_y}),$$

by the Decomposition Theorem. Since $\pi_*(\mathbb{1}_V[1])|_{C_0}$ is one-dimensional and $\mathcal{IC}(\mathbb{1}_{C_y})|_{C_0}$ is one-dimensional, it follows that $\mathcal{IC}(\mathcal{E}_{C_y})|_{C_0} = 0$.

Thus, the geometric multiplicity matrix is

	$\mathcal{S}(\mathbb{1}_{C_0})$	$\mathcal{S}(\mathbb{1}_{C_y})$	$\mathcal{S}(\mathcal{E}_{C_y})$
$\mathcal{IC}(\mathbb{1}_{C_0})$	1	0	0
$\mathcal{IC}(\mathbb{1}_{C_y})$	-1	1	0
$\mathcal{IC}(\mathcal{E}_{C_y})$	0	0	1

and the normalised geometric multiplicity matrix is

	$\mathbb{1}^\natural_{C_0}$	$\mathbb{1}^\natural_{C_y}$	$\mathcal{E}^\natural_{C_y}$
$\mathbb{1}^\sharp_{C_0}$	1	0	0
$\mathbb{1}^\sharp_{C_y}$	1	1	0
$\mathcal{E}^\sharp_{C_y}$	0	0	1

2.2.3. *Cuspidal support decomposition and Fourier transform.* Up to conjugation, $\widehat{G} = \mathrm{SL}(2,\mathbb{C})$ admits exactly two cuspidal Levi subgroups: $\widehat{G}$ itself and $\widehat{T} = \mathrm{GL}(1)$. Thus,

$$\mathsf{Per}_{H_\lambda}(V_\lambda) = \mathsf{Per}_{H_\lambda}(V_\lambda)_{\widehat{T}} \oplus \mathsf{Per}_{H_\lambda}(V_\lambda)_{\widehat{G}}.$$

Simple objects in these two subcategories are listed below.

$\mathsf{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{\widehat{T}/\mathrm{iso}}$	$\mathsf{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{\widehat{G}/\mathrm{iso}}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	
$\mathcal{IC}(\mathbb{1}_{C_y})$	$\mathcal{IC}(\mathcal{E}_{C_y})$

The Fourier transform is given on simply objects by:

$$\begin{array}{rcl} \mathsf{Ft} : \mathsf{Per}_{H_\lambda}(V_\lambda) & \longrightarrow & \mathsf{Per}_{H_\lambda}(V_\lambda^*) \\ \mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_0^*}) = \mathcal{IC}(\mathbb{1}_{C_y^t}) \\ \mathcal{IC}(\mathbb{1}_{C_y}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_y^*}) = \mathcal{IC}(\mathbb{1}_{C_0^t}) \\ \mathcal{IC}(\mathcal{E}_{C_y}) & \mapsto & \mathcal{IC}(\mathcal{E}_{C_0^*}) = \mathcal{IC}(\mathcal{E}_{C_y^t}) \end{array}$$

2.2.4. *Equivariant local systems on the regular conormal bundle.* The regular conormal bundle $T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}}$ decomposes into two H_λ orbits

$$T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}} = T^*_{C_0}(V_\lambda)_{\mathrm{reg}} \bigsqcup T^*_{C_y}(V_\lambda)_{\mathrm{reg}}$$

given by

$$T^*_{C_0}(V_\lambda)_{\mathrm{reg}} = \left\{ \begin{pmatrix} 0 & y \\ y' & 0 \end{pmatrix} \;\middle|\; \begin{array}{c} y = 0 \\ y' \neq 0 \end{array} \right\}, \qquad T^*_{C_y}(V_\lambda)_{\mathrm{reg}} = \left\{ \begin{pmatrix} 0 & y \\ y' & 0 \end{pmatrix} \;\middle|\; \begin{array}{c} y \neq 0 \\ y' = 0 \end{array} \right\}.$$

We remark that

$$T^*_{C_0}(V_\lambda)_{\mathrm{reg}} = T^*_{C_0}(V_\lambda)_{\mathrm{sreg}} = C_0 \times C_0^* \qquad \text{and} \qquad T^*_{C_y}(V_\lambda)_{\mathrm{reg}} = T^*_{C_y}(V_\lambda)_{\mathrm{sreg}} = C_y \times C_y^*.$$

These components are H_λ-orbits, so every H-equivariant perverse sheaf on $T^*_H(V)_{\mathrm{reg}}$ is a standard sheaf shifted to degree 1. The equivariant fundamental groups are both given by

$$A_C^{\mathrm{mic}} = \pi_1(T^*_C(V_\lambda), (x,\xi))_{Z_{H_\lambda}(x,\xi)^0} = \pi_0(Z_{H_\lambda}(x,\xi)) = Z(\widehat{G}) \cong \{\pm 1\}.$$

Let $\mathbb{1}_{\mathcal{O}_\psi}$ be the constant local system on $T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}$ and let $\mathcal{E}_{\mathcal{O}_\psi}$ be the non-trivial H-equivariant local system on $T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}$. Then

$$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_j}) = \mathcal{S}(\mathbb{1}_{\mathcal{O}_j}) \qquad \text{and} \qquad \mathcal{IC}(\mathcal{E}_{\mathcal{O}_j}) = \mathcal{S}(\mathcal{E}_{\mathcal{O}_j}).$$

In summary,

$$\mathsf{Loc}_H(T^*_{C_0}(V)_{\mathrm{sreg}})^{\mathrm{simple}}_{/\mathrm{iso}} = \{\mathbb{1}_{\mathcal{O}_0},\ \mathcal{E}_{\mathcal{O}_0}\}$$

and

$$\mathsf{Loc}_H(T^*_{C_y}(V)_{\mathrm{sreg}})^{\mathrm{simple}}_{/\mathrm{iso}} = \left\{\mathbb{1}_{\mathcal{O}_y},\ \mathcal{E}_{\mathcal{O}_y}\right\}.$$

TABLE 2.2.5.1. ${}^p\mathsf{Ev} : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}})$ on simple objects, for $\lambda : W_F \to {}^LG$ given at the beginning of Section 2.

$$\begin{array}{ccc} \mathsf{Per}_H(V) & \xrightarrow{{}^p\mathsf{Ev}} & \mathsf{Per}_H(T^*_H(V)_{\mathrm{reg}}) \\ \mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) \\ \mathcal{IC}(\mathbb{1}_{C_y}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_y}) \\ \mathcal{IC}(\mathcal{E}_{C_y}) & \mapsto & \mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}) \end{array}$$

$\mathcal{P}$	$\mathsf{Evs}_{C_0}\mathcal{P}$	$\mathsf{Evs}_{C_y}\mathcal{P}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$+$	0
$\mathcal{IC}(\mathbb{1}_{C_y})$	0	$+$
$\mathcal{IC}(\mathcal{E}_{C_y})$	$-$	$-$

2.2.5. *Vanishing cycles of perverse sheaves.* The functor ${}^p\mathsf{Ev} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(T^*_H(V)_{\mathrm{reg}})$ is given on simple objects in Table 2.2.5.1. The lower part uses the identification of local systems on the regular conormal with representations of the corresponding equivariant fundamental groups, so each $\mathsf{Evs}_C\,\mathcal{P}$ is given as a character of A^{mic}_C.

We now explain the computations behind Table 2.2.5.1.

(a) From [7, Theorem 5.8.1] it follows immediately that

Now we need to prove this by hand, unfortunately.

$$\begin{array}{rclrcl} {}^p\mathsf{Ev}_{C_y}\,\mathcal{IC}(\mathbb{1}_{C_y}) & = & \mathbb{1}_{\mathcal{O}_y}[1] & {}^p\mathsf{Ev}_{C_y}\,\mathcal{IC}(\mathcal{E}_{C_y}) & = & \mathcal{E}_{\mathcal{O}_y}[1] \\ {}^p\mathsf{Ev}_{C_y}\,\mathcal{IC}(\mathbb{1}_{C_0}) & = & 0 & {}^p\mathsf{Ev}_{C_0}\,\mathcal{IC}(\mathbb{1}_{C_0}) & = & \mathbb{1}_{\mathcal{O}_0}[0]. \end{array}$$

It only remains, therefore, to determine ${}^p\mathsf{Ev}_{C_0}\,\mathcal{IC}(\mathbb{1}_{C_y})$ and ${}^p\mathsf{Ev}_{C_0}\,\mathcal{IC}(\mathcal{E}_{C_y})$.

(b) Since $\mathcal{IC}(\mathbb{1}_{C_y}) = \mathbb{1}_V[1]$, we have

$$\mathsf{Ev}_{C_0}\,\mathcal{IC}(\mathbb{1}_{C_y}) = R\Phi_{yy'}(\mathbb{1}_V[1] \boxtimes \mathbb{1}_{C_0^*})|_{T^*_C(V)_{\mathrm{reg}}}.$$

As $\mathbb{1}_V \boxtimes \mathbb{1}_{C_0^*} = \mathbb{1}_{V\times C_0^*}$ is a local system and the function $(y, y') \mapsto yy'$ is smooth on $V \times C_0^*$, it follows [10, Exposé XIII, Reformulation 2.1.5] that

$$\mathsf{Ev}_{C_0}\,\mathcal{IC}(\mathbb{1}_{C_y}) = 0.$$

Note that C_0^* specifically excludes the locus $y' = 0$, which is where the singularities would be.

(c) We now consider the case of $\mathcal{IC}(\mathcal{E}_{C_y})$, using the proper double cover $\pi : V \to V$, already used in Section 2.2.2. Recall that

$$\pi_*(\mathbb{1}_V[1]) = \mathcal{IC}(\mathbb{1}_{C_y}) \oplus \mathcal{IC}(\mathcal{E}_{C_y}).$$

Since Ev is exact by [7, Proposition 5.3.1],

$$\mathsf{Ev}_{C_0}\,\pi_*(\mathbb{1}_V[1]) = \mathsf{Ev}_{C_0}\,\mathcal{IC}(\mathbb{1}_{C_y}) \oplus \mathsf{Ev}_{C_0}\,\mathcal{IC}(\mathcal{E}_{C_y}).$$

We have just seen that $\mathsf{Ev}_{C_0}\,\mathcal{IC}(\mathbb{1}_{C_y}) = 0$, so

$$\mathsf{Ev}_{C_0}\,\mathcal{IC}(\mathcal{E}_{C_y}) = \mathsf{Ev}_{C_0}\,\pi_*(\mathbb{1}_V[1]).$$

By [7, Lemma 5.4.2],

$$\mathsf{Ev}_{C_0}\,\pi_*(\mathbb{1}_V[1]) = \pi_!\left(R\Phi_{y^2y'}(\mathbb{1}_{V\times C_0^*}[1])|_{T^*_C(V)_{\pi\text{-reg}}}\right).$$

Since π is an isomorphism on $T^*_C(V)_{\pi\text{-reg}}$,

$$\mathsf{Ev}_{C_0}\,\pi_*(\mathbb{1}_V[1]) = R\Phi_{y^2y'}(\mathbb{1}_{V\times C_0^*}[1])|_{T^*_C(V)_{\mathrm{reg}}}.$$

Now,

$$R\Phi_{y^2y'}(\mathbb{1}_{V\times C_0^*}[1]) = \pi'_!\mathbb{1}_{C_0\times C_0^*}[1],$$

where $\pi' : C_0^* \to C_0^*$ is the double cover $y' \mapsto y'^2$. Note that

$$\pi'_!\mathbb{1}_{C_0\times C_0^*}[1] = \pi'_!\mathbb{1}_{\mathcal{O}_0}[1].$$

By the Decomposition Theorem,

$$\pi'_!\mathbb{1}_{\mathcal{O}_0}[1] = \mathbb{1}_{\mathcal{O}_0}[1] \oplus \mathcal{E}_{\mathcal{O}_0}[1],$$

where $\mathcal{E}_{\mathcal{O}_0}$ is the non-trivial equivariant local system on $\mathcal{O}_0$ introduced in Section 2.2.4, which is the associated to the double cover arising from taking $\sqrt{y'}$ over $\mathcal{O}_0$. Therefore,

$${}^p\mathsf{Ev}_{C_0}\mathcal{IC}(\mathcal{E}_{C_y}) = \mathcal{E}_{\mathcal{O}_0}[1].$$

This completes the calculation of ${}^p\mathsf{Ev} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(T^*(V)_{\mathrm{reg}})$ on simple objects, as displayed in Table 2.2.5.1.

2.2.6. *Normalization of Ev and the twisting local system.* From Table 2.2.5.1 we see that the twisting local system $\mathcal{T}$ is trivial in this case, so ${}^p\mathsf{NEv} = {}^p\mathsf{Ev}$.

2.2.7. *Fourier transform and vanishing cycles.* Having found ${}^p\mathsf{Ev} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(T^*_H(V)_{\mathrm{reg}})$ on simple objects, we also know ${}^p\mathsf{Ev}^* : \mathsf{Per}_H(V^*) \to \mathsf{Per}_H(T^*_H(V^*)_{\mathrm{reg}})$ on simple objects. We use this and the coincidence of ${}^p\mathsf{Ev}$ with ${}^p\mathsf{NEv}$, in the table below.

$\mathsf{Per}_H(V)$	$\xrightarrow{{}^p\mathsf{Ev}}$	$\mathsf{Per}_H(T^*_H(V)_{\mathrm{reg}})$	$\xrightarrow{a_*}$	$\mathsf{Per}_H(T^*_H(V^*)_{\mathrm{reg}})$	$\xleftarrow{\mathsf{Ev}^*}$	$\mathsf{Per}_H(V^*)$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0^*})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathbb{1}_{C_0^*})$
$\mathcal{IC}(\mathbb{1}_{C_y})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_y})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_y^*})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathbb{1}_{C_y^*})$
$\mathcal{IC}(\mathcal{E}_{C_y})$	$\mapsto$	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})$	$\mapsto$	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}_y^*}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0^*})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathcal{E}_{C_0^*})$

Since the map from the first to the fourth column is the Fourier transform, this verifies (21).

2.2.8. *Arthur sheaves.*

Arthur sheaf	packet sheaves	coronal sheaves
$\mathcal{A}_{C_0}$	$\mathcal{IC}(\mathbb{1}_{C_0}) \oplus$	$\mathcal{IC}(\mathcal{E}_{C_y})$
$\mathcal{A}_{C_y}$	$\mathcal{IC}(\mathbb{1}_{C_y}) \oplus \mathcal{IC}(\mathcal{E}_{C_y})$	

2.3. **Adams-Barbasch-Vogan packets.**

2.3.1. *Admissible representations versus equivariant perverse sheaves.* Vogan's bijection for $\lambda : W_F \to {}^LG$ chosen at the beginning of Section 2 is given by the following table:

$\mathsf{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{/\mathrm{iso}}$	$\Pi_{\mathrm{pure},\lambda}(G/F)$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$(\pi(\phi_0), 0)$
$\mathcal{IC}(\mathbb{1}_{C_y})$	$(\pi(\phi_1), 0)$
$\mathcal{IC}(\mathcal{E}_{C_y})$	$(\pi(\phi_1, -), 1)$

The base points for H-orbits in $T^*_H(V)_{\mathrm{reg}}$ determined by the Arthur parameters ψ_0 and ψ_1 are:

$$(x_{\psi_0}, \xi_{\psi_0}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in T^*_{C_0}(V_\lambda)_{\mathrm{reg}}, \qquad (x_{\psi_1}, \xi_{\psi_1}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in T^*_{C_y}(V_\lambda)_{\mathrm{reg}}.$$

2.3.2. *ABV-packets.* Using the bijection of Section 2.3.1, the vanishing cycles calculations of Section 2.2.5, and the definition of ABV-packets from [7], we find ABV-packets for G representations with infinitesimal parameter $\lambda : W_F \to {}^LG$ from Section 2.1.1:

$$\Pi^{\mathrm{ABV}}_{\mathrm{pure},\psi_0}(G/F) \quad = \quad \left\{ \begin{array}{c} [\pi(\phi_0), 0] \\ [\pi(\phi_1, -), 1] \end{array} \right\}, \qquad \Pi^{\mathrm{ABV}}_{\mathrm{pure},\psi_1}(G/F) \quad = \quad \left\{ \begin{array}{c} [\pi(\phi_1, +), 0] \\ [\pi(\phi_1, -), 1] \end{array} \right\}.$$

We see that all pure Arthur packets are Adams-Barbasch-Vogan packets simply by comparing this with Section 2.1.4. In this example, all the strata in V are of Arthur type, so all ABV-packets are Arthur packets.

2.3.3. *Stable invariant distributions and their endoscopic transfer.* In Section 2.1.6 we recalled the coefficient appearing in the invariant distributions $\eta_{\psi,s}$ attached to $\psi \in Q_\lambda({}^LG)$ and $s \in Z_{\widehat{G}}(\psi)$. Using Section 2.2.5, compare $\langle ss_\psi, [\pi, \delta] \rangle_\psi$ with trace $\mathsf{Ev}_\psi \mathcal{P}(\pi, \delta)(ss_\psi)$. This proves (27) and therefore establishes [7, Conjecture 2], in this case:

$$\eta_{\psi,s} = \eta^{\mathsf{NEv}}_{\psi,s},$$

for $\psi \in Q_\lambda({}^LG)$ and $s \in Z_{\widehat{G}}(\psi)$.

Also recall from Section 2.1.6 that the only endoscopic group relevant to ψ_0 and ψ_1 is G_0.

2.3.4. *Kazhdan-Lusztig conjecture.* Using the bijection of Section 2.3.1 we may compare the multiplicity matrix from Section 2.1.3 with the normalised geometric multiplicity matrix from Section 2.2.2:

$$m_{\mathrm{rep}} = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array} \right), \qquad m'_{\mathrm{geo}} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array} \right).$$

Since ${}^t m_{\mathrm{rep}} = m'_{\mathrm{geo}}$, this confirms the Kazhdan-Lusztig conjecture as it applies to representations with infinitesimal parameter $\lambda : W_F \to {}^LG$ given in Section 2.1.1.

2.3.5. *Aubert duality and Fourier transform.* Use Vogan's bijection from Section 2.3.1 to compare Aubert duality from Section 2.1.5 with the Fourier transform from Section 2.2.3 to verify (30).

Compare the twisting characters χ_ψ of A_ψ from Section 2.1.5 with the restriction $\mathcal{T}_\psi$ to $T^*_{C_\psi}(V)_{\mathrm{reg}}$ of the local system $\mathcal{T}_\psi$ from Section 2.2.7 to verify (31).

2.4. **Endoscopy and equivariant restriction of perverse sheaves.** The material of Section 0.4 is trivial in this example, since $Z_{\widehat{G}}(\psi) = Z(\widehat{G})$.

3. PGL(4) shallow representations

This example illustrates the utility of [7, Theorem 3.1.1] and the significance of the decomposition of $\lambda(\mathrm{Fr})$ into hyperbolic and elliptic parts. Here, the calculation of the Arthur packets for certain non-tempered representations of PGL(4) is reduced to the calculation of certain unipotent representations of SL(2). This example also example concerns a case when $H^1(F, G_{\mathrm{ad}}) \to H^1(F, \mathrm{Aut}(G))$ is surjective but not injective.

Set $G = \mathrm{PGL}(4)$ over F and suppose q is odd. So, $\widehat{G} = \mathrm{SL}(4)$ and ${}^LG = \mathrm{SL}(4) \times W_F$. In this case, $H^1(F, G) = H^1(F, G_{\mathrm{ad}}) \cong \mathrm{Irrep}(\mu_4)$, so there are four isomorphism classes of inner forms of G, each one pure. However, G has only three forms, up to isomorphism: the split group $G_0 = G$ itself, an anisotropic form G_1, and a non-quasi-split form G_2 with

a proper minimal Levi. In fact, the outer automorphism of G induces an action of order 2 on $H^1(F,G)$, and the orbits of this action correspond exactly to the image of $H^1(F,G)$ in $H^1(F,\mathrm{Aut}(G))$. The map $H^1(F,G_{\mathrm{ad}}) \to H^1(F,\mathrm{Aut}(G))$ from isomorphism classes of inner form of G to isomorphism classes of forms of G is given by: $0 \mapsto G_0$, $1 \mapsto G_1$, $2 \mapsto G_2$ and $3 \mapsto G_1$, where the notation refers to an identification of $\mathrm{Irrep}(\mu_4)$ with $\mathbb{Z}/4\mathbb{Z}$.

Let E be the Galois closure of the ramified extension $F(\sqrt[q+1]{\varpi})$. Then E is the compositum of an unramified quadratic extension of F and the totally ramified extension $F(\sqrt[q+1]{\varpi})$; now $\mathrm{Gal}(E/F)$ is the dihedral group with generators σ, τ, where σ has order 2 and τ has order $q+1$ and $\sigma\tau\sigma = \tau^{-1} = \tau^q$. Consider the representation $\varrho : \mathrm{Gal}(E/F) \to \mathrm{SL}(2,\mathbb{C})$ defined by

$$\sigma \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tau \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix},$$

where $\zeta \in C$ is a fixed primitive $q+1$-th root of unity. Let $\rho : W_F \to \mathrm{SL}(2,\mathbb{C})$ be the composition of $W_F \to \Gamma_F \to \mathrm{Gal}(E/F)$ with ϱ. Define $\lambda : W_F \to {}^LG = \mathrm{SL}(4) \times W_F$ by

$$\lambda(w) := \rho(w) \otimes \nu_2(d_w).$$

Thus, if $w|_E = \sigma$ then

$$\lambda(w) = \begin{pmatrix} 0 & 0 & |w|^{1/2} & 0 \\ 0 & 0 & 0 & |w|^{-1/2} \\ -|w|^{1/2} & 0 & 0 & 0 \\ 0 & -|w|^{-1/2} & 0 & 0 \end{pmatrix}$$

while if $w|_E = \tau$ then

$$\lambda(w) = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & \zeta^{-1} & 0 \\ 0 & 0 & 0 & \zeta^{-1} \end{pmatrix}.$$

3.1. **Arthur packets.**

3.1.1. *Parameters.* There are two Langlands parameters with infinitesimal parameter λ, each of Arthur type:

$$\begin{aligned} \phi_0(w,x) &:= \rho(w) \otimes \nu_2(d_w), & \phi_1(w,x) &:= \rho(w) \otimes \nu_2(x) \\ \psi_0(w,x,y) &:= \rho(w) \otimes \nu_2(y), & \psi_1(w,x,y) &:= \rho(w) \otimes \nu_2(x). \end{aligned}$$

Note that ψ_0 and ψ_1 are Aubert dual.

3.1.2. *L-packets.* There are 5 admissible representations of the three forms G_0, G_1 and G_2, with infinitesimal parameter λ. In order to list them, we start with the component groups of $\phi \in P_\lambda({}^LG)$. First, note that

$$Z_{\widehat{G}}(\lambda) = \left\{ \begin{pmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_1 & 0 \\ 0 & 0 & 0 & s_2 \end{pmatrix} \;\middle|\; s_1 s_2 = \pm 1 \right\} \cong \mathrm{GL}(1) \times \mu_2,$$

under the isomorphism $s \mapsto (s_1, s_1 s_2)$. Then

$$A_{\phi_0} = \pi_0(Z_{\widehat{G}}(\phi_0)) = \pi_0(Z_{\widehat{G}}(\lambda)) \cong \mu_2 \qquad \text{and} \qquad A_{\phi_1} = \pi_0(Z_{\widehat{G}}(\phi_1)) = \pi_0(Z(\widehat{G})) \cong \mu_4.$$

Following our convention, we write $+$ and $-$ for the trivial and non-trivial characters of μ_2, respectively; the characters of μ_4 will be labeled by $+1$, -1, $+i$ and $-i$. The admissible representations for the Langlands parameters ϕ_0 and ϕ_1 fall into L-packets for the three forms of G (up to isomorphism) as follows:

$$\begin{array}{rclrcl}
\Pi_{\phi_0}(G_0(F)) & = & \{\pi(\phi_0,+)\} & \Pi_{\phi_1}(G_0(F)) & = & \{\pi(\phi_1,+1)\} \\
\Pi_{\phi_0}(G_1(F)) & = & \emptyset & \Pi_{\phi_1}(G_1(F)) & = & \{\pi(\phi_1,+i)\} \\
 & & & & = & \{\pi(\phi_1,-i)\} \\
\Pi_{\phi_0}(G_2(F)) & = & \{\pi(\phi_0,-)\} & \Pi_{\phi_1}(G_2(F)) & = & \{\pi(\phi_1,-1)\}.
\end{array}$$

However, $\Pi_{\mathrm{pure},\lambda}(G/F)$ consists of 6 representations of 4 pure rational forms of G:

$$\Pi_{\mathrm{pure},\phi_0}(G/F) = \left\{ \ [\pi(\phi_0,+),0], \quad [\pi(\phi_0,-),2] \ \right\},$$

and

$$\Pi_{\mathrm{pure},\phi_1}(G/F) = \left\{ \ [\pi(\phi_1,+1),0], \quad [\pi(\phi_1,+i),1], \quad [\pi(\phi_1,-1),2], \quad [\pi(\phi_1,-i),3] \ \right\}.$$

In other words, when passing from the four equivalence classes of pure rational forms $[\delta] \in H^1(F,G)$ to the three isomorphism classes of forms of G, two representations collapse to one, namely, $[\pi(\phi_1,+i),1]$ and $[\pi(\phi_1,-i),3]$ map to the same admissible representation of $G_1(F)$.

Ahmed, could you write a few words here about the representations $\pi(\phi_0,+)$, $\pi(\phi_0,-)$, $\pi(\phi_1,+1)$, $\pi(\phi_1,-1)$, $\pi(\phi_1,+i)$ and $\pi(\phi_1,-i)$?

3.1.3. *Multiplicities in standard modules.*

	$\pi(\phi_0,+)$	$\pi(\phi_0,-)$	$\pi(\phi_1,+1)$	$\pi(\phi_1,-1)$	$\pi(\phi_1,+i)$	$\pi(\phi_1,-i)$
$M(\phi_0,+1)$	1	0	1	0	0	0
$M(\phi_0,-1)$	0	1	0	1	0	0
$M(\phi_1,+1)$	0	0	1	0	0	0
$M(\phi_1,-1)$	0	0	0	1	0	0
$M(\phi_1,+i)$	0	0	0	0	1	0
$M(\phi_1,-i)$	0	0	0	0	0	1

3.1.4. *Arthur packets.* The component groups A_{ψ_0} and A_{ψ_1} are both $Z(\widehat{G})$, canonically. Arthur packets for rational forms G_0, G_1 and G_2 of G are

$$\begin{array}{rclrcl}
\Pi_{\psi_0}(G_0(F)) & = & \{\pi(\phi_0,+)\} & \Pi_{\psi_1}(G_0(F)) & = & \{\pi(\phi_1,+1)\} \\
\Pi_{\psi_0}(G_1(F)) & = & \{\pi(\phi_1,+i)\} & \Pi_{\psi_1}(G_1(F)) & = & \{\pi(\phi_1,+i)\} \\
 & = & \{\pi(\phi_1,-i)\} & & = & \{\pi(\phi_1,-i)\} \\
\Pi_{\psi_0}(G_2(F)) & = & \{\pi(\phi_0,-)\} & \Pi_{\psi_1}(G_2(F)) & = & \{\pi(\phi_1,-1)\}
\end{array}$$

The pure Arthur packets for ψ_0 and ψ_1 are

$$\Pi_{\mathrm{pure},\psi_0}(G/F) = \left\{ \ [\pi(\phi_0,+),0], \quad [\pi(\phi_0,-),2], \quad [\pi(\phi_1,+i),1], \quad [\pi(\phi_1,-i),3] \ \right\},$$

and

$$\Pi_{\mathrm{pure},\psi_1}(G/F) = \left\{ \ [\pi(\phi_1,+1),0], \quad [\pi(\phi_1,+i),1], \quad [\pi(\phi_1,-1),2], \quad [\pi(\phi_1,-i),3] \ \right\}.$$

For later reference, we break these pure Arthur packets apart into packet and coronal representations:

pure Arthur packets	pure L-packet representations	coronal representations
$\Pi_{\mathrm{pure},\psi_0}(G/F)$	$[\pi(\phi_0,+),0]$, $[\pi(\phi_0,-),2]$,	$[\pi(\phi_1,+i),1]$, $[\pi(\phi_i,-i),3]$
$\Pi_{\mathrm{pure},\psi_1}(G/F)$	$[\pi(\phi_1,+1),0]$, $[\pi(\phi_1,+i),1]$, $[\pi(\phi_1,-i),3]$,	$[\pi(\phi_1,-1),2]$

3.1.5. *Aubert duality.* Aubert duality for admissible representations of $G_0(F)$ with infinitesimal parameter λ is given by the following table.

π	$\hat{\pi}$
$\pi(\phi_0,+)$	$\pi(\phi_1,+1)$
$\pi(\phi_1,+1)$	$\pi(\phi_0,+)$

Aubert duality for $G_1(F)=G_3(F)$ is given by the following table.

π	$\hat{\pi}$
$\pi(\phi_1,+i)=\pi(\phi_1,-i)$	$\pi(\phi_1,+i)=\pi(\phi_1,-i)$

Aubert duality for $G_2(F)$ is given by the following table.

π	$\hat{\pi}$
$\pi(\phi_0,-)$	$\pi(\phi_1,-1)$
$\pi(\phi_1,-1)$	$\pi(\phi_0,-)$

The twisting characters χ_{ψ_0} and χ_{ψ_1} are trivial.

3.1.6. *Stable distributions and endoscopy.* The coefficients $\langle a_s a_\psi, (\pi,\delta)\rangle_\psi$ appearing in the invariant distributions $\eta_{\psi,s}$ (16) are given by the following list, in which $s\in A_\psi = Z(\widehat{G})\cong\mu_4$.

$$\begin{aligned}
\eta_{\psi_0}=\eta_{\psi_0,1} &= [\pi(\phi_0,+),0]+[\pi(\phi_0,-),2]+[\pi(\phi_1,+i),1]+[\pi(\phi_1,-i),3]\\
\eta_{\psi_0,-1} &= [\pi(\phi_0,+),0]+[\pi(\phi_0,-),2]-[\pi(\phi_1,+i),1]-[\pi(\phi_1,-i),3]\\
\eta_{\psi_0,i} &= [\pi(\phi_0,+),0]-[\pi(\phi_0,-),2]+i[\pi(\phi_1,+i),1]-i[\pi(\phi_1,-i),3]\\
\eta_{\psi_0,-i} &= [\pi(\phi_0,+),0]-[\pi(\phi_0,-),2]-i[\pi(\phi_1,+i),1]+i[\pi(\phi_1,-i),3]
\end{aligned}$$

and

$$\begin{aligned}
\eta_{\psi_1}=\eta_{\psi_1,1} &= [\pi(\phi_1,+1),0]-[\pi(\phi_1,-i),1]+[\pi(\phi_1,-1),2]-[\pi(\phi_1,-i),3]\\
\eta_{\psi_1,-1} &= [\pi(\phi_1,+1),0]-[\pi(\phi_1,-i),1]-[\pi(\phi_1,-1),2]+[\pi(\phi_1,-i),3]\\
\eta_{\psi_1,i} &= [\pi(\phi_1,+1),0]+[\pi(\phi_1,-i),1]+i[\pi(\phi_1,-1),2]+i[\pi(\phi_1,-i),3]\\
\eta_{\psi_1,-i} &= [\pi(\phi_1,+1),0]+[\pi(\phi_1,-i),1]-i[\pi(\phi_1,-1),2]-i[\pi(\phi_1,-i),3].
\end{aligned}$$

Since $A_{\psi_0}=Z(\widehat{G})$ and $A_{\psi_1}=Z(\widehat{G})$, the only endoscopic groups relevant to these Arthur parameters are $G=G_0$, G_1 and G_2.

3.2. **Vanishing cycles of perverse sheaves.**

3.2.1. *Vogan variety and orbit duality.* The Vogan variety V_λ and its dual V_λ^* may both be deduced from the conormal bundle

$$T_H^*(V)=\left\{\begin{pmatrix}0&y&0&0\\y'&0&0&0\\0&0&0&y\\0&0&y'&0\end{pmatrix}\mid yy'=0\right\}$$

on which $H:=Z_{\widehat{G}}(\lambda)\cong \mathrm{GL}(1)\times\nu_2$ acts by

$$\begin{pmatrix}s_1&0&0&0\\0&s_2&0&0\\0&0&s_1&0\\0&0&0&s_2\end{pmatrix}\cdot\begin{pmatrix}0&y&0&0\\y'&0&0&0\\0&0&0&y\\0&0&y'&0\end{pmatrix}=\begin{pmatrix}0&s_1s_2^{-1}y&0&0\\s_1^{-1}s_2y'&0&0&0\\0&0&0&s_1s_2^{-1}y\\0&0&s_1^{-1}s_2y'&0\end{pmatrix}.$$

Recall that $s_1 s_2 = \pm 1$, so $s_1 s_2^{-1} = \pm s_1^2$. From this we see the stratification of V into H-orbits and the duality on those orbits is exactly as in Section 2.2.1.

We now use [7, Theorem 3.1.1] to replace $\lambda : W_F \to {}^L G$ with an unramified infinitesimal parameter $\lambda_{\mathrm{nr}} : W_F \to {}^L G_\lambda$ of a split group G_λ such that $\lambda_{\mathrm{nr}}(\mathrm{Fr})$ is hyperbolic. The hyperbolic part of $\lambda(\mathrm{Fr})$ is $s_\lambda \times 1$ with

$$s_\lambda = \rho(1) \otimes \nu_2(\mathrm{Fr}) = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & q^{-1/2} & 0 & 0 \\ 0 & 0 & q^{1/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix}$$

while the elliptic part of $\lambda(\mathrm{Fr})$ is $t_\lambda \times \mathrm{Fr}$ with

$$t_\lambda = \rho(\mathrm{Fr}) \otimes \nu_2(1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then

$$J_\lambda := Z_{\widehat{G}}(\lambda|_{I_F}, s_\lambda) = \left\{ \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \;\middle|\; \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1 \right\} \cong \mathrm{SL}(2) \times \mu_2$$

under the isomorphism $\mathrm{diag}(h, h) \mapsto (h', \det h)$ where $h' = h$ if $\det h = 1$ and $h' = ih$ if $\det h = -1$. Therefore, $G_\lambda = \mathrm{PGL}(2)$ and $\lambda_{\mathrm{nr}} : W_F \to {}^L G_\lambda$ is given by

Check this, again.

$$\lambda_{\mathrm{nr}}(w) = \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}.$$

Now

$$H_{\lambda_{\mathrm{nr}}} = Z_{\widehat{G}_\lambda}(\lambda_{\mathrm{nr}}) = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \;\middle|\; t \neq 0 \right\} \cong \mathrm{GL}(1)$$

and

$$V_{\lambda_{\mathrm{nr}}} = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \;\middle|\; y \right\} \cong \mathbb{A}^1$$

with $H_{\lambda_{\mathrm{nr}}}$-action

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & t^2 y \\ 0 & 0 \end{pmatrix}.$$

This brings us back to Section 2.2.1. We will freely use notation from there, below. The H_λ-action on $V_{\lambda_{\mathrm{nr}}}$ is given by

$$(t, \pm 1) : \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \pm t^2 y \\ 0 & 0 \end{pmatrix}.$$

From this we see that every H_λ-orbit in $V_{\lambda_{\mathrm{nr}}}$ coincides with a $H_{\lambda_{\mathrm{nr}}}$ orbit in $V_{\lambda_{\mathrm{nr}}}$.

3.2.2. *Equivariant perverse sheaves on Vogan variety.* With reference to [7, Theorem 3.1.1] we have

$$\begin{array}{ccccc} \mathsf{Rep}(A_\lambda) & \longrightarrow & \mathsf{Per}_{H_\lambda}(V_\lambda) & \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} & \mathsf{Per}_{H_{\lambda_{\mathrm{nr}}}}(V_{\lambda_{\mathrm{nr}}}) \\ \| & & \| & & \| \\ \mathsf{Rep}(\mu_2) & & \mathsf{Per}_{\mathrm{GL}(1)\times\mu_2}(\mathbb{A}^1) & & \mathsf{Per}_{\mathrm{GL}(1)}(\mathbb{A}^1) \end{array}$$

The image of the trivial representation $+$ of μ_2 under the functor $\mathsf{Rep}(A_\lambda) \to \mathsf{Per}_{H_\lambda}(V_\lambda)$ is the trivial local system on V, denoted here by $(+)_V$ to emphasise its genesis; image of the non-trivial irreducible representation $-$ of μ_2 under the functor $\mathsf{Rep}(A_\lambda) \to \mathsf{Per}_{H_\lambda}(V_\lambda)$ will likewise be denoted by $(-)_V$.

To find the simple objects in $\mathsf{Per}_H(V)$, we begin with the equivariant perverse sheaves on H-orbits in V.

- C_0: The equivariant fundamental group of C_0 is $A_{C_0} = \pi_0(H) \cong \mu_2$. Let us write $\mathbb{1}^+_{C_0}$ and $\mathbb{1}^-_{C_0}$ for the local systems corresponding to the trivial and non-trivial representations of A_{C_0}, respectively. Note that, under the forgetful functor $\mathsf{Loc}_H(C_0) \to \mathsf{Loc}_{H_{\mathrm{nr}}}(C_0)$, these both map to $\mathbb{1}_{C_0}$, the constant sheaf on C_0.
- C_y: The equivariant fundamental group of C_y is $A_{C_y} = Z(\widehat{G}) \cong \mu_4$. Let us write $\mathbb{1}^+_{C_y}$ and $\mathbb{1}^-_{C_y}$ for the equivariant local systems on C_y that correspond to the trivial $+1$ and order-2 characters -1 of A_{C_y}, respectively; these both map to $\mathbb{1}_{C_y}$ under $\mathsf{Loc}_H(C_y) \to \mathsf{Loc}_{H_{\mathrm{nr}}}(C_y)$. We write $\mathcal{E}^+_{C_y}$ and $\mathcal{E}^-_{C_y}$ for the equivariant local systems on C_y that correspond to the order-4 characters $+i$ and $-i$, respectively, of A_{C_y}; these both map to $\mathcal{E}_{C_y}$ under $\mathsf{Loc}_H(C_y) \to \mathsf{Loc}_{H_{\mathrm{nr}}}(C_y)$.

Therefore, the six simple objects in $\mathsf{Per}_H(V)$ are given by:

$$\mathsf{Per}_H(V)^{\mathrm{simple}}_{/\mathrm{iso}} = \left\{ \begin{array}{lll} \mathcal{IC}(\mathbb{1}^+_{C_0}), & \mathcal{IC}(\mathbb{1}^+_{C_y}), & \mathcal{IC}(\mathcal{E}^+_{C_y}) \\ \mathcal{IC}(\mathbb{1}^-_{C_0}), & \mathcal{IC}(\mathbb{1}^-_{C_y}), & \mathcal{IC}(\mathcal{E}^-_{C_y}) \end{array} \right\}.$$

On simple objects, the functor $\mathsf{Rep}(A_\lambda) \to \mathsf{Per}_{H_\lambda}(V)$ is given by $(+)_V[1] = \mathcal{IC}(\mathbb{1}^+_{C_y})$ and $(-)_V[1] = \mathcal{IC}(\mathbb{1}^-_{C_y})$; the functor $\mathsf{Per}_{H_\lambda}(V) \to \mathsf{Per}_{H_{\lambda_{\mathrm{nr}}}}(V_{\lambda_{\mathrm{nr}}})$ is given by $\mathcal{IC}(\mathbb{1}^\pm_{C_0}) \mapsto \mathcal{IC}(\mathbb{1}_{C_0})$ and $\mathcal{IC}(\mathbb{1}^\pm_{C_y}) \mapsto \mathcal{IC}(\mathbb{1}_{C_y})$ and $\mathcal{IC}(\mathcal{E}^\pm_{C_y}) \mapsto \mathcal{IC}(\mathcal{E}_{C_y})$; the functor $\mathsf{Per}_{H_{\lambda_{\mathrm{nr}}}}(V_{\lambda_{\mathrm{nr}}}) \to \mathsf{Per}_{H_\lambda}(V)$ is given by $\mathcal{IC}(\mathbb{1}_{C_0}) \mapsto \mathcal{IC}(\mathbb{1}^+_{C_0}) \oplus \mathcal{IC}(\mathbb{1}^-_{C_0})$ and $\mathcal{IC}(\mathbb{1}_{C_y}) \mapsto \mathcal{IC}(\mathbb{1}^+_{C_y}) \oplus \mathcal{IC}(\mathbb{1}^-_{C_y})$ and $\mathcal{IC}(\mathcal{E}_{C_y}) \mapsto \mathcal{IC}(\mathcal{E}^+_{C_y}) \oplus \mathcal{IC}(\mathcal{E}^-_{C_y})$.

From this we find the stalks of the simple objects in $\mathsf{Per}_H(V)$.

$\mathcal{P}$	$\mathcal{P}\vert_{C_0}$	$\mathcal{P}\vert_{C_{+1}}$
$\mathcal{IC}(\mathbb{1}^+_{C_0})$	$\mathbb{1}^+_{C_0}[0]$	0
$\mathcal{IC}(\mathbb{1}^-_{C_0})$	$\mathbb{1}^-_{C_0}[0]$	0
$\mathcal{IC}(\mathbb{1}^+_{C_{+1}})$	$\mathbb{1}^+_{C_0}[1]$	$\mathbb{1}^+_{C_{+1}}[1]$
$\mathcal{IC}(\mathbb{1}^-_{C_{+1}})$	$\mathbb{1}^-_{C_0}[1]$	$\mathbb{1}^-_{C_{+1}}[1]$
$\mathcal{IC}(\mathcal{E}^+_{C_{+1}})$	0	$\mathcal{E}^+_{C_{+1}}[1]$
$\mathcal{IC}(\mathcal{E}^-_{C_{+1}})$	0	$\mathcal{E}^-_{C_{+1}}[1]$

This gives us the normalised geometric multiplicity matrix:

	$(\mathbb{1}^+_{C_0})^\natural$	$(\mathbb{1}^-_{C_0})^\natural$	$(\mathbb{1}^+_{C_1})^\natural$	$(\mathbb{1}^-_{C_1})^\natural$	$(\mathcal{E}^+_{C_1})^\natural$	$(\mathcal{E}^-_{C_1})^\natural$
$(\mathbb{1}^+_{C_0})^\sharp$	1	0	0	0	0	0
$(\mathbb{1}^-_{C_0})^\sharp$	0	1	0	0	0	0
$(\mathbb{1}^+_{C_1})^\sharp$	1	0	1	0	0	0
$(\mathbb{1}^-_{C_1})^\sharp$	0	1	0	1	0	0
$(\mathcal{E}^+_{C_1})^\sharp$	0	0	0	0	1	0
$(\mathcal{E}^-_{C_1})^\sharp$	0	0	0	0	0	1

3.2.3. *Cuspidal support decomposition and Fourier transform.* The cuspidal support decomposition respects the functors appearing in [7, Theorem 3.1.1], so the results here follow from Section 2.2.3. Specifically, we have

$$\mathsf{Per}_{H_\lambda}(V_\lambda) = \mathsf{Per}_{H_\lambda}(V_\lambda)_{\widehat{T}} \oplus \mathsf{Per}_{H_\lambda}(V_\lambda)_{\widehat{G}},$$

where the simple objects in these summand categories are given here.

$\mathsf{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{\widehat{T}/\text{iso}}$	$\mathsf{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{\widehat{G}/\text{iso}}$
$\mathcal{IC}(\mathbb{1}^+_{C_0})$	
$\mathcal{IC}(\mathbb{1}^-_{C_0})$	
$\mathcal{IC}(\mathbb{1}^+_{C_y})$	$\mathcal{IC}(\mathcal{E}^+_{C_y})$
$\mathcal{IC}(\mathbb{1}^-_{C_y})$	$\mathcal{IC}(\mathcal{E}^-_{C_y})$

Since the diagram

$$\begin{tikzcd}
\mathsf{Rep}(A_\lambda) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathsf{Per}_{H_\lambda}(V_\lambda) \arrow[r, shift left, "\pi^*"] \arrow[d, "\mathsf{Ft}"] & \mathsf{Per}_{H_{\lambda_{\mathrm{nr}}}}(V_{\lambda_{\mathrm{nr}}}) \arrow[l, shift left, "\pi_*"] \arrow[d, "\mathsf{Ft}"] \\
\mathsf{Rep}(A_\lambda) \arrow[r] & \mathsf{Per}_{H_\lambda}(V^*_\lambda) \arrow[r, shift left, "\pi^*"] & \mathsf{Per}_{H_{\lambda_{\mathrm{nr}}}}(V^*_{\lambda_{\mathrm{nr}}}) \arrow[l, shift left, "\pi_*"]
\end{tikzcd}$$

commutes, the Fourier transform is given on simple objects as follows.

$$\begin{array}{rcl}
\mathsf{Ft} : \mathsf{Per}_{H_\lambda}(V_\lambda) & \longrightarrow & \mathsf{Per}_{H_\lambda}(V^*_\lambda) \\
\mathcal{IC}(\mathbb{1}^+_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}^+_{C_0^*}) = \mathcal{IC}(\mathbb{1}^+_{C_1^t}) \\
\mathcal{IC}(\mathbb{1}^-_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}^-_{C_0^*}) = \mathcal{IC}(\mathbb{1}^-_{C_1^t}) \\
\mathcal{IC}(\mathbb{1}^+_{C_y}) & \mapsto & \mathcal{IC}(\mathbb{1}^+_{C_y^*}) = \mathcal{IC}(\mathbb{1}^+_{C_0^t}) \\
\mathcal{IC}(\mathbb{1}^-_{C_y}) & \mapsto & \mathcal{IC}(\mathbb{1}^-_{C_y^*}) = \mathcal{IC}(\mathbb{1}^-_{C_0^t}) \\
\mathcal{IC}(\mathcal{E}^+_{C_y}) & \mapsto & \mathcal{IC}(\mathcal{E}^+_{C_0^*}) = \mathcal{IC}(\mathcal{E}^+_{C_y^t}) \\
\mathcal{IC}(\mathcal{E}^-_{C_y}) & \mapsto & \mathcal{IC}(\mathcal{E}^-_{C_0^*}) = \mathcal{IC}(\mathcal{E}^-_{C_y^t})
\end{array}$$

3.2.4. *Equivariant perverse sheaves on the regular conormal bundle.* Recall that H_λ orbits coincide with $H_{\lambda_{\mathrm{nr}}}$-orbits. The following diagram commutes:

$$\begin{tikzcd}
\mathsf{Rep}(A_\lambda) \arrow[r] \arrow[d, equal] & \mathsf{Per}_{H_\lambda}(C^*) \arrow[r, shift left, "\pi^*"] \arrow[d] & \mathsf{Per}_{H_{\lambda_{\mathrm{nr}}}}(C^*) \arrow[l, shift left, "\pi_*"] \arrow[d] \\
\mathsf{Rep}(A_\lambda) \arrow[r] \arrow[d, equal] & \mathsf{Per}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{sreg}}) \arrow[r, shift left, "\pi^*"] & \mathsf{Per}_{H_{\lambda_{\mathrm{nr}}}}(T^*_C(V_{\lambda_{\mathrm{nr}}})_{\mathrm{sreg}}) \arrow[l, shift left, "\pi_*"] \\
\mathsf{Rep}(A_\lambda) \arrow[r] & \mathsf{Per}_{H_\lambda}(C) \arrow[u] \arrow[r, shift left, "\pi^*"] & \mathsf{Per}_{H_{\lambda_{\mathrm{nr}}}}(C) \arrow[u] \arrow[l, shift left, "\pi_*"]
\end{tikzcd}$$

We now describe the fundamental groups and associated equivariant local systems on the strongly regular conormal bundle $T^*_H(V)_{\mathrm{sreg}}$. For the computation of the functor $\mathsf{Ev} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(T^*_H(V)_{\mathrm{reg}})$ in Section 3.2.5 we will need to know the effect of pullback along the bundle map $T^*_H(V)_{\mathrm{reg}} \to V$, so we also give that information below.

C_0: We choose a base point for $T^*_{C_0}(V)_{\mathrm{sreg}}$:

$$(x_0, \xi_0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $A_{(x_0,\xi_0)} = Z(\widehat{G}) \cong \mu_4$ and the bundle maps induce the following homomorphisms of fundamental groups:

$$\mu_2 \cong A_{x_0} \twoheadleftarrow A_{(x_0,\xi_0)} \xrightarrow{\mathrm{id}} A_{\xi_0} \cong \mu_4$$

Now label local systems on $T^*_{C_0}(V)_{\mathrm{sreg}}$ according to the following chart, which lists the corresponding characters of $A_{(x_0,\xi_0)}$ using the convention for characters of μ_4 from Section 3.1.2.

$\mathsf{Loc}_H(T^*_{C_0}(V)_{\mathrm{sreg}})$:	$\mathbb{1}^+_{\mathcal{O}_0}$	$\mathbb{1}^-_{\mathcal{O}_0}$	$\mathcal{E}^+_{\mathcal{O}_0}$	$\mathcal{E}^-_{\mathcal{O}_0}$
$\mathsf{Rep}(A_{(x_0,\xi_0)})$:	$+1$	-1	$+i$	$-i$

Pullback of equivariant local systems along the bundle map $T^*_{C_0}(V)_{\mathrm{sreg}} \to C_0$ is given on simple objects by:

$$\begin{array}{ccc} \mathsf{Loc}_H(C_0) & \to & \mathsf{Loc}_H(T^*_{C_0}(V)_{\mathrm{sreg}}) \\ \mathbb{1}^\pm_{C_0} & \mapsto & \mathbb{1}^\pm_{\mathcal{O}_0} \\ & & \mathcal{E}^\pm_{\mathcal{O}_0} \end{array}$$

C_y: We choose a base point for $T^*_{C_y}(V)_{\mathrm{sreg}}$:

$$(x_1, \xi_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $A_{(x_1,\xi_1)} = Z(\widehat{G}) \cong \mu_4$ and the bundle maps induce the following homomorphisms of fundamental groups:

$$\mu_4 \cong A_{x_1} \xleftarrow{\mathrm{id}} A_{(x_1,\xi_1)} \longrightarrow A_{\xi_1} \cong \mu_2$$

Now label local systems on $T^*_{C_y}(V)_{\mathrm{sreg}}$ according to the following chart, which lists the corresponding characters of $A_{(x_1,\xi_1)}$ using the convention for characters

TABLE 3.2.5.1. ${}^p\mathsf{Ev} : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}})$ on simple objects, for $\lambda : W_F \to {}^LG$ given at the beginning of Section 3.

$$\begin{array}{ccc}
\mathsf{Per}_H(V) & \xrightarrow{{}^p\mathsf{Ev}} & \mathsf{Per}_H(T^*_{H_\lambda}(V)_{\mathrm{reg}}) \\
\mathcal{IC}(\mathbb{1}^+_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}^+_{\mathcal{O}_0}) \\
\mathcal{IC}(\mathbb{1}^-_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}^-_{\mathcal{O}_0}) \\
\mathcal{IC}(\mathbb{1}^+_{C_y}) & \mapsto & \mathcal{IC}(\mathbb{1}^+_{\mathcal{O}_y}) \\
\mathcal{IC}(\mathbb{1}^-_{C_y}) & \mapsto & \mathcal{IC}(\mathbb{1}^-_{\mathcal{O}_y}) \\
\mathcal{IC}(\mathcal{E}^+_{C_y}) & \mapsto & \mathcal{IC}(\mathcal{E}^+_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}^+_{\mathcal{O}_0}) \\
\mathcal{IC}(\mathcal{E}^-_{C_y}) & \mapsto & \mathcal{IC}(\mathcal{E}^-_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}^-_{\mathcal{O}_0})
\end{array}$$

$\mathcal{P}$	$\mathsf{Evs}_{C_0}\,\mathcal{P}$	$\mathsf{Evs}_{C_1}\,\mathcal{P}$
$\mathcal{IC}(\mathbb{1}^+_{C_0})$	$+1$	0
$\mathcal{IC}(\mathbb{1}^-_{C_0})$	-1	0
$\mathcal{IC}(\mathbb{1}^+_{C_y})$	0	$+1$
$\mathcal{IC}(\mathbb{1}^-_{C_y})$	0	-1
$\mathcal{IC}(\mathcal{E}^+_{C_y})$	$+i$	$+i$
$\mathcal{IC}(\mathcal{E}^-_{C_y})$	$-i$	$-i$

of μ_4 from Section 3.1.2.

$$\boxed{\begin{array}{rcccc}
\mathsf{Loc}_H(T^*_{C_y}(V)_{\mathrm{sreg}}) : & \mathbb{1}^+_{\mathcal{O}_y} & \mathbb{1}^-_{\mathcal{O}_y} & \mathcal{E}^+_{\mathcal{O}_y} & \mathcal{E}^-_{\mathcal{O}_y} \\
\mathsf{Rep}(A_{(x_1,\xi_1)}) : & +1 & -1 & +i & -i
\end{array}}$$

Pullback of equivariant local systems along the bundle map $T^*_{C_y}(V)_{\mathrm{sreg}} \to C_y$ is given on simple objects by:

$$\begin{array}{ccc}
\mathsf{Loc}_H(C_y) & \to & \mathsf{Loc}_H(T^*_{C_y}(V)_{\mathrm{sreg}}) \\
\mathbb{1}^\pm_{C_y} & \mapsto & \mathbb{1}^\pm_{\mathcal{O}_y} \\
\mathcal{E}^\pm_{C_y} & \mapsto & \mathcal{E}^\pm_{\mathcal{O}_y}
\end{array}$$

3.2.5. *Vanishing cycles of perverse sheaves.* Table 3.2.5.1 gives the functor $\mathsf{Ev} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(T^*_H(V)_{\mathrm{reg}})$ on simple objects. These calculations follow from Table 2.2.5.1.

3.2.6. *Normalization of Ev and the twisting local system.* From Table 3.2.5.1 we see that the twisting local system $\mathcal{T}$ is trivial in this case, so ${}^p\mathsf{NEv} = {}^p\mathsf{Ev}$.

3.2.7. *Vanishing cycles and Fourier transform.* Comparing the table below with $\mathsf{Ft} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(V^*)$ from Section 3.2.3 verifies (21) in this example.

$$\begin{array}{ccccccc}
\mathsf{Per}_H(V) & \xrightarrow{{}^p\mathsf{Ev}} & \mathsf{Per}_H(T^*_{H_\lambda}(V)_{\mathrm{reg}}) & \xrightarrow{a_*} & \mathsf{Per}_H(T^*_{H_\lambda}(V^*)_{\mathrm{reg}}) & \xleftarrow{\mathsf{Ev}^*} & \mathsf{Per}_H(V^*) \\
\mathcal{IC}(\mathbb{1}^\pm_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}^\pm_{\mathcal{O}_0}) & \mapsto & \mathcal{IC}(\mathbb{1}^\pm_{\mathcal{O}^*_0}) & \leftarrow\!\shortmid & \mathcal{IC}(\mathbb{1}^\pm_{C^*_0}) \\
\mathcal{IC}(\mathbb{1}^\pm_{C_y}) & \mapsto & \mathcal{IC}(\mathbb{1}^\pm_{\mathcal{O}_y}) & \mapsto & \mathcal{IC}(\mathbb{1}^\pm_{\mathcal{O}^*_y}) & \leftarrow\!\shortmid & \mathcal{IC}(\mathbb{1}^\pm_{C^*_y}) \\
\mathcal{IC}(\mathcal{E}^\pm_{C_y}) & \mapsto & \mathcal{IC}(\mathcal{E}^\pm_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}^\pm_{\mathcal{O}_0}) & \mapsto & \mathcal{IC}(\mathcal{E}^\pm_{\mathcal{O}^*_y}) \oplus \mathcal{IC}(\mathcal{E}^\pm_{\mathcal{O}^*_0}) & \leftarrow\!\shortmid & \mathcal{IC}(\mathcal{E}^\pm_{C^*_0})
\end{array}$$

3.2.8. *Arthur sheaves.*

Arthur sheaf	packet sheaves	coronal sheaves
$\mathcal{A}_{C_0}$	$\mathcal{IC}(\mathbb{1}^+_{C_0}) \oplus \mathcal{IC}(\mathbb{1}^-_{C_0}) \oplus$	$\mathcal{IC}(\mathcal{E}^+_{C_y}) \oplus \mathcal{IC}(\mathcal{E}^-_{C_y})$
$\mathcal{A}_{C_y}$	$\mathcal{IC}(\mathbb{1}^+_{C_y}) \oplus \mathcal{IC}(\mathbb{1}^-_{C_y}) \oplus \mathcal{IC}(\mathcal{E}^+_{C_y}) \oplus \mathcal{IC}(\mathcal{E}^-_{C_y})$	

3.3. **Adams-Barbasch-Vogan packets.**

3.3.1. *Admissible representations versus perverse sheaves.*

$\mathsf{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}}$	$\Pi_{\text{pure},\lambda}(G/F)$
$\mathcal{IC}(\mathbb{1}^+_{C_0})$	$(\pi(\phi_0,+),0)$
$\mathcal{IC}(\mathbb{1}^-_{C_0})$	$(\pi(\phi_0,-),2)$
$\mathcal{IC}(\mathbb{1}^+_{C_y})$	$(\pi(\phi_1,+1),0)$
$\mathcal{IC}(\mathbb{1}^-_{C_y})$	$(\pi(\phi_1,-1),2)$
$\mathcal{IC}(\mathcal{E}^+_{C_y})$	$(\pi(\phi_1,+i),1)$
$\mathcal{IC}(\mathcal{E}^-_{C_y})$	$(\pi(\phi_1,-i),3)$

3.3.2. *ABV-packets.*

ABV-packets	pure L-packet representations	coronal representations
$\Pi^{\text{ABV}}_{\text{pure},\phi_0}(G/F):$	$[\pi(\phi_0,+),0],\ [\pi(\phi_0,-),2]$	$[\pi(\phi_1,+i),1],\ [\pi(\phi_1,-i),3]$
$\Pi^{\text{ABV}}_{\text{pure},\phi_1}(G/F):$	$[\pi(\phi_1,+1),0],\ [\pi(\phi_1,+i),1],\ [\pi(\phi_1,-1),2],\ [\pi(\phi_1,-i),3]$	

3.3.3. *Stable distributions and endoscopic transfer.*

$$\begin{aligned}
\eta^{\mathsf{NEv}}_{\psi_0} = \eta^{\mathsf{NEv}}_{\psi_0,1} &= [\pi(\phi_0,+),0] + [\pi(\phi_0,-),2] + [\pi(\phi_1,+i),1] + [\pi(\phi_1,-i),3] \\
\eta^{\mathsf{NEv}}_{\psi_0,-1} &= [\pi(\phi_0,+),0] + [\pi(\phi_0,-),2] - [\pi(\phi_1,+i),1] - [\pi(\phi_1,-i),3] \\
\eta^{\mathsf{NEv}}_{\psi_0,i} &= [\pi(\phi_0,+),0] - [\pi(\phi_0,-),2] + i[\pi(\phi_1,+i),1] - i[\pi(\phi_1,-i),3] \\
\eta^{\mathsf{NEv}}_{\psi_0,-i} &= [\pi(\phi_0,+),0] - [\pi(\phi_0,-),2] - i[\pi(\phi_1,+i),1] + i[\pi(\phi_1,-i),3]
\end{aligned}$$

$$\begin{aligned}
\eta^{\mathsf{NEv}}_{\psi_1} = \eta^{\mathsf{NEv}}_{\psi_1,1} &= [\pi(\phi_1,1),0] - [\pi(\phi_1,i),1] + [\pi(\phi_1,-1),2] - [\pi(\phi_1,-i),3] \\
\eta^{\mathsf{NEv}}_{\psi_1,-1} &= [\pi(\phi_1,1),0] - [\pi(\phi_1,i),1] - [\pi(\phi_1,-1),2] + [\pi(\phi_1,-i),3] \\
\eta^{\mathsf{NEv}}_{\psi_1,i} &= [\pi(\phi_1,1),0] + [\pi(\phi_1,i),1] + i[\pi(\phi_1,-1),2] + i[\pi(\phi_1,-i),3] \\
\eta^{\mathsf{NEv}}_{\psi_1,-i} &= [\pi(\phi_1,1),0] + [\pi(\phi_1,i),1] - i[\pi(\phi_1,-1),2] - i[\pi(\phi_1,-i),3]
\end{aligned}$$

Comparing with Section 3.1.6 proves (28).

3.3.4. *Kazhdan-Lusztig conjecture.* From Section 3.1.3 we find the multiplicity matrix:

$$m_{\text{rep}} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and from Section 3.2.2 we find the normalised geometric multiplicity matrix

$$m'_{\mathrm{geo}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $m^t_{\mathrm{rep}} = m'_{\mathrm{geo}}$, this proves the Kazhdan-Lusztig conjecture in this case.

Notice that

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and compare with Section 2.3.4.

3.3.5. *Aubert duality and Fourier transform.* To verify (30), use Vogan's bijection from Section 3.3.1 to compare Aubert duality from Section 3.1.5 with the Fourier transform from Section 3.2.3

To verify (31), observe that the twisting characters χ_ψ of A_ψ from Section 3.1.5 are trivial, as are the local systems $\mathcal{T}_\psi$ from Section 3.2.7.

3.4. **Endoscopy and equivariant restriction of perverse sheaves.** The material of Section 0.4 is trivial in this example, since $Z_{\widehat{G}}(\psi) = Z(\widehat{G})$.

4. SO(5) unipotent representations, regular infinitesimal parameter

In this example, of the four Langlands parameters with infinitesimal parameter λ below, only two are of Arthur type. Accordingly, we find two ABV-packet that are not Arthur packets.

Let $G = \mathrm{SO}(5)$, so $\widehat{G} = \mathrm{Sp}(4)$ and ${}^LG = \widehat{G} \times W_F$. As in the cases above,

$$H^1(F, G) = H^1(F, G_{\mathrm{ad}}) = H^1(F, \mathrm{Aut}(G)) \cong \mathbb{Z}/2\mathbb{Z},$$

so there are two isomorphism classes of rational forms of G, each pure. We will use the notation $G_0 = G$ and G_1 for the non-quasisplit form of SO(5) given by the quadratic form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -\varepsilon\varpi & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & \varpi & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\lambda : W_F \to \widehat{G}$ be the unramified homomorphism

$$\lambda(\mathrm{Fr}) = \begin{pmatrix} |w|^{3/2} & 0 & 0 & 0 \\ 0 & |w|^{1/2} & 0 & 0 \\ 0 & 0 & |w|^{-1/2} & 0 \\ 0 & 0 & 0 & |w|^{-3/2} \end{pmatrix}.$$

Here and below we use the symplectic form $\langle x, y\rangle = {}^t x J y$ with matrix J given by $J_{ij} = (-1)^j \delta_{5-i,j}$ to determine a representation of $\widehat{G} = \mathrm{Sp}(4)$.

Although this example exhibits some interesting geometric phenomena, there is still no interesting endoscopy here. Nevertheless, this example will be important later when we consider other groups for which SO(5) is an endoscopic group.

4.1. **Arthur packets.**

4.1.1. *Parameters.* Up to $Z_{\widehat{G}}(\lambda)$-conjugation, there are four Langlands parameters with infinitesimal parameter λ:

$$\begin{aligned}
\phi_0(w,x) &= \nu_4(d_w) = \lambda(w),\\
\phi_1(w,x) &= \nu_2^2(d_w)\otimes\nu_2(x) = \left(\begin{array}{cc|cc} |w|x_{11} & |w|x_{11} & 0 & 0\\ |w|x_{21} & |w|x_{22} & 0 & 0\\ \hline 0 & 0 & |w|^{-1}x_{11} & |w|^{-1}x_{12}\\ 0 & 0 & |w|^{-1}x_{11} & |w|^{-1}x_{12}\end{array}\right),\\
\phi_2(w,x) &= \nu_2^3(d_w)\oplus\nu_2(x) = \left(\begin{array}{c|cc|c} |w|^{3/2} & 0 & 0 & 0\\ \hline 0 & x_{11} & x_{12} & 0\\ 0 & x_{21} & x_{22} & 0\\ \hline 0 & 0 & 0 & |w|^{-3/2}\end{array}\right),\\
\phi_3(w,x) &= \nu_4(x),
\end{aligned}$$

where $\nu_4 : \mathrm{SL}(2) \to \mathrm{Sp}(4)$ is the irreducible 4-dimensional representation of SL(2). Of the four Langlands parameters ϕ_0, ϕ_1, ϕ_2 and ϕ_3, only ϕ_0 and ϕ_3 are of Arthur type; define

$$\psi_0(w,x,y) \quad := \quad \nu_4(y), \qquad \text{and} \qquad \psi_3(w,x,y) \quad := \quad \nu_4(x).$$

4.1.2. *L-packets.* The component groups A_{ϕ_0} and A_{ϕ_1} are trivial, while the component groups A_{ϕ_2} and A_{ϕ_3} each have order two, being canonically isomorphic to $Z(\widehat{G})$. Therefore, the representations in play in this example are:

$$\begin{array}{rclrcl}
\Pi_{\phi_0}(G_0(F)) &=& \{\pi(\phi_0)\}, & \Pi_{\phi_0}(G_1(F)) &=& \emptyset,\\
\Pi_{\phi_1}(G_0(F)) &=& \{\pi(\phi_1)\}, & \Pi_{\phi_1}(G_1(F)) &=& \emptyset,\\
\Pi_{\phi_2}(G_0(F)) &=& \{\pi(\phi_2,+)\}, & \Pi_{\phi_2}(G_1(F)) &=& \{\pi(\phi_2,-)\},\\
\Pi_{\phi_3}(G_0(F)) &=& \{\pi(\phi_3,+)\}, & \Pi_{\phi_3}(G_1(F)) &=& \{\pi(\phi_3,-)\}.
\end{array}$$

Of the four admissible representations of $G(F)$ with infinitesimal parameter λ, only $\pi(\phi_3,+)$ is tempered – this is the Steinberg representation. The representation $\pi(\phi_1)$ (resp. $\pi(\phi_2,+)$) is denoted by $L(\nu\zeta\,\mathrm{St}_{\mathrm{GL}(2)})$ (resp. $L(\nu^{3/2}\zeta, \zeta\,\mathrm{St}_{\mathrm{SO}(3)})$) with $\zeta = 1$ in [18]. When arranged into pure packets, we get

$$\begin{array}{rcl}
\Pi_{\mathrm{pure},\phi_0}(G/F) &=& \{[\pi(\phi_0),0]\}\\
\Pi_{\mathrm{pure},\phi_1}(G/F) &=& \{[\pi(\phi_1),0]\}\\
\Pi_{\mathrm{pure},\phi_2}(G/F) &=& \{[\pi(\phi_2,+),0],\ [\pi(\phi_2,-),1]\}\\
\Pi_{\mathrm{pure},\phi_3}(G/F) &=& \{[\pi(\phi_3,+),0],\ [\pi(\phi_3,-),1]\}
\end{array}$$

4.1.3. *Multiplicities in standard modules.* The standard module $M(\phi_1)$ (resp. $M(\phi_2,+)$) is denoted by $\nu\zeta\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes 1$ (resp. $\nu^{3/2}\zeta \rtimes \mathrm{St}_{\mathrm{SO}(3)}$) with $\zeta = 1$ in [18]. The following table

may be deduced from [18, Proposition 3.3].

	$\pi(\phi_0)$	$\pi(\phi_1)$	$\pi(\phi_2,+)$	$\pi(\phi_3,+)$	$\pi(\phi_2,-)$	$\pi(\phi_3,-)$
$M(\phi_0)$	1	1	1	1	0	0
$M(\phi_1)$	0	1	0	1	0	0
$M(\phi_2,+)$	0	0	1	1	0	0
$M(\phi_3,+)$	0	0	0	1	0	0
$M(\phi_2,-)$	0	0	0	0	1	1
$M(\phi_3,-)$	0	0	0	0	0	1

4.1.4. *Arthur packets.* The Arthur packets for these representations are

$$\begin{array}{rclrcl} \Pi_{\psi_0}(G_0(F)) & = & \{\pi(\phi_0)\}, & \Pi_{\psi_0}(G_1(F)) & = & \{\pi(\phi_2,-)\}, \\ \Pi_{\psi_3}(G_0(F)) & = & \{\pi(\phi_3,+)\}, & \Pi_{\psi_3}(G_1(F)) & = & \{\pi(\phi_3,-)\}. \end{array}$$

When arranged into pure packets, we get

$$\begin{array}{rcl} \Pi_{\mathrm{pure},\psi_0}(G/F) & = & \{[\pi(\phi_0),0],\ [\pi(\phi_2,-),1]\}, \\ \Pi_{\mathrm{pure},\psi_3}(G/F) & = & \{[\pi(\phi_3,+),0],\ [\pi(\phi_3,-),1]\}. \end{array}$$

4.1.5. *Aubert duality.* Aubert duality for $G_0(F)$ and $G_1(F)$ are given by the following table.

π	$\hat{\pi}$
$\pi(\phi_0)$	$\pi(\phi_3,+)$
$\pi(\phi_1)$	$\pi(\phi_2,+)$
$\pi(\phi_2,+)$	$\pi(\phi_1)$
$\pi(\phi_3,+)$	$\pi(\phi_0)$
$\pi(\phi_2,-)$	$\pi(\phi_3,-)$
$\pi(\phi_3,-)$	$\pi(\phi_2,-)$

The twisting characters χ_{ψ_0} and χ_{ψ_1} are trivial.

4.1.6. *Stable distributions and endoscopic transfer.* For $s \in Z(\widehat{G}) \cong \mu_2$, the virtual representations $\eta_{\psi_0,s}$ and $\eta_{\psi_3,s}$ are given by

$$\begin{array}{rcl} \eta_{\psi_0,1} & = & [\pi(\phi_0),0] + [\pi(\phi_2,-),1] \\ \eta_{\psi_0,-1} & = & [\pi(\phi_0),0] - [\pi(\phi_2,-),1] \end{array}$$

and

$$\begin{array}{rcl} \eta_{\psi_3,1} & = & [\pi(\phi_3,+),0] - [\pi(\phi_3,-),1] \\ \eta_{\psi_3,-1} & = & [\pi(\phi_3,+),0] + [\pi(\phi_3,-),1]. \end{array}$$

There are no endoscopic groups relevant to ψ_0 or ψ_3 other than G_0.

4.2. **Vanishing cycles of perverse sheaves.**

4.2.1. *Vogan variety and orbit duality.* Now

$$H := Z_{\widehat{G}}(\lambda) = \left\{ \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \ \middle|\ \begin{array}{c} t_1 \neq 0 \\ t_2 \neq 0 \end{array} \right\}.$$

The Vogan varieties V and V^* are given by

$$V = \left\{ \begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid u, x \right\}, \qquad V^* = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ u' & 0 & 0 & 0 \\ 0 & x' & 0 & 0 \\ 0 & 0 & u' & 0 \end{pmatrix} \mid u', x' \right\}.$$

The action of H on $T^*(V)$ is given by

$$\begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} : \begin{pmatrix} 0 & u & 0 & 0 \\ u' & 0 & x & 0 \\ 0 & x' & 0 & u \\ 0 & 0 & u' & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & t_1 t_2^{-1} u & 0 & 0 \\ t_1^{-1} t_2 u' & 0 & t_2^2 x & 0 \\ 0 & t_2^{-2} x' & 0 & t_1 t_2^{-1} u \\ 0 & 0 & t_1^{-1} t_2 u' & 0 \end{pmatrix}.$$

The conormal bundle is

$$T^*_{H_\lambda}(V_\lambda) \cong \left\{ \begin{pmatrix} 0 & u & 0 & 0 \\ u' & 0 & x & 0 \\ 0 & x' & 0 & u \\ 0 & 0 & u' & 0 \end{pmatrix} \mid \begin{matrix} uu' = 0 \\ xx' = 0 \end{matrix} \right\}.$$

Now V is stratified into the following H-orbits:

$$C_0 := \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \qquad C_3 := \left\{ \begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid \begin{matrix} u \neq 0 \\ x \neq 0 \end{matrix} \right\},$$

and

$$C_u := \left\{ \begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid u \neq 0 \right\}, \qquad C_x := \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid x \neq 0 \right\}.$$

The dual orbits in V^* are

$$C_0^* = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ u' & 0 & 0 & 0 \\ 0 & x' & 0 & 0 \\ 0 & 0 & u' & 0 \end{pmatrix} \mid \begin{matrix} u' \neq 0 \\ x' \neq 0 \end{matrix} \right\}, \qquad C_{ux}^* = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

and

$$C_u^* = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x' & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid x' \neq 0 \right\}, \qquad C_x^* = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ u' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u' & 0 \end{pmatrix} \mid u' \neq 0 \right\}.$$

The following diagram gives the closure relations for these orbits.

$$\begin{tikzcd}[column sep=small]
 & C_{ux} = \hat{C}_0 & & \dim = 2 & & C_0^* = C_{ux}^t & \\
C_u = \hat{C}_x \arrow[ur] & & C_x = \hat{C}_u \arrow[ul] & \dim = 1 & C_u^* = C_x^t \arrow[ur] & & C_x^* = C_u^t \arrow[ul] \\
 & C_0 = \hat{C}_{ux} \arrow[ul] \arrow[ur] & & \dim = 0 & & C_{ux}^* = C_0^t \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

4.2.2. *Equivariant perverse sheaves.* The equivariant fundamental groups for C_0 and C_u are trivial, so they each carry only one equivariant local system, denoted by $\mathbb{1}_{C_0}$ and $\mathbb{1}_{C_u}$, respectively. The equivariant fundamental groups for C_x and C_{ux} have order two, so they each carry two equivariant local systems, denoted by $\mathbb{1}_{C_x}$, $\mathcal{L}_{C_x}$, $\mathbb{1}_{C_{ux}}$ and $\mathcal{L}_{C_{ux}}$. Thus,

$$\mathsf{Per}_H(V)^{\text{simple}}_{/\text{iso}} = \{\mathcal{IC}(\mathbb{1}_{C_0}),\ \mathcal{IC}(\mathbb{1}_{C_u}),\ \mathcal{IC}(\mathbb{1}_{C_x}),\ \mathcal{IC}(\mathbb{1}_{C_{ux}}),\ \mathcal{IC}(\mathcal{L}_{C_x}),\ \mathcal{IC}(\mathcal{L}_{C_{ux}})\}.$$

The following table describes these perverse sheaves on H-orbits in V.

$\mathcal{P}$	$\mathcal{P}\vert_{C_0}$	$\mathcal{P}\vert_{C_u}$	$\mathcal{P}\vert_{C_x}$	$\mathcal{P}\vert_{C_{u,x}}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathbb{1}_{C_0}[0]$	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_u})$	$\mathbb{1}_{C_0}[1]$	$\mathbb{1}_{C_u}[1]$	0	0
$\mathcal{IC}(\mathbb{1}_{C_x})$	$\mathbb{1}_{C_0}[1]$	0	$\mathbb{1}_{C_x}[1]$	0
$\mathcal{IC}(\mathbb{1}_{C_{ux}})$	$\mathbb{1}_{C_0}[2]$	$\mathbb{1}_{C_u}[2]$	$\mathbb{1}_{C_x}[2]$	$\mathbb{1}_{C_{ux}}[2]$
$\mathcal{IC}(\mathcal{L}_{C_x})$	0	0	$\mathcal{L}_{C_x}[1]$	0
$\mathcal{IC}(\mathcal{L}_{C_{ux}})$	0	0	$\mathcal{L}_{C_x}[2]$	$\mathcal{L}_{C_{ux}}[2]$

We now explain how to make these calculations.

(a) For the first four rows in the table above, those that deal with $\mathcal{IC}(\mathbb{1}_C)$, it is sufficient to observe that the closure $\overline{C}$ of each strata C is smooth, hence the sheaf $\mathbb{1}_{\overline{C}}[\dim(C)]$ is perverse.

(b) For the remaining two rows, those that deal with $\mathcal{IC}(\mathcal{L}_C)$, we observe that the closure $\overline{C}$ of the strata C admits a finite equivarient double cover $\pi : \widetilde{C} \to \overline{C}$ by taking $\sqrt{x}$. Because $\widetilde{C}$ is smooth, the sheaf $\mathbb{1}_{\widetilde{C}}[\dim(C)]$ is perverse. The decomposition theorem for finite maps of perverse sheaves yields $\pi_!(\mathbb{1}_{\widetilde{C}}[\dim(C)]) = \mathcal{IC}(\mathbb{1}_{\overline{C}}) \oplus \mathcal{IC}(\mathcal{L}_{\overline{C}})$. Proper base change, the decomposition theorem for finite étale maps, and our earlier computations for $\mathcal{IC}(\mathbb{1}_{\overline{C}})$ then allows us to readily compute the stalks of $\mathcal{IC}(\mathcal{L}_{\overline{C}})$.

From this, we easily find the normalised geometric multiplicity matrix

	$\mathbb{1}^\natural_{C_0}$	$\mathbb{1}^\natural_{C_u}$	$\mathbb{1}^\natural_{C_x}$	$\mathbb{1}^\natural_{C_{ux}}$	$\mathcal{L}^\natural_{C_x}$	$\mathcal{L}^\natural_{C_{ux}}$
$\mathbb{1}^\sharp_{C_0}$	1	0	0	0	0	0
$\mathbb{1}^\sharp_{C_u}$	1	1	0	0	0	0
$\mathbb{1}^\sharp_{C_x}$	1	0	1	0	0	0
$\mathbb{1}^\sharp_{C_{ux}}$	1	1	1	1	0	0
$\mathcal{L}^\sharp_{C_x}$	0	0	0	0	1	0
$\mathcal{L}^\sharp_{C_{ux}}$	0	0	0	0	1	1

4.2.3. *Cuspidal support decomposition and Fourier transform.* Up to conjugation, $\widehat{G} = \mathrm{Sp}(4)$ admits exactly two cuspidal Levi subgroups: $\widehat{M} = \mathrm{Sp}(2) \times \mathrm{GL}(1)$ and $\widehat{T} = \mathrm{GL}(1) \times \mathrm{GL}(1)$.

$$\mathsf{Per}_{H_\lambda}(V_\lambda) = \mathsf{Per}_{H_\lambda}(V_\lambda)_{\widehat{T}} \oplus \mathsf{Per}_{H_\lambda}(V_\lambda)_{\widehat{M}}.$$

Simple objects in these two subcategories are listed below.

$\mathsf{Per}_H(V)_{\check{T}}$	$\mathsf{Per}_H(V)_{\check{M}}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	
$\mathcal{IC}(\mathbb{1}_{C_u})$	
$\mathcal{IC}(\mathbb{1}_{C_x})$	$\mathcal{IC}(\mathcal{L}_{C_x})$
$\mathcal{IC}(\mathbb{1}_{C_{ux}})$	$\mathcal{IC}(\mathcal{L}_{C_{ux}})$

The Fourier transform is given as follows:

$$\begin{array}{rcl}
\mathsf{Ft} : \mathsf{Per}_H(V) & \longrightarrow & \mathsf{Per}_H(V^*) \\
\mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_0^*}) = \mathcal{IC}(\mathbb{1}_{C_{ux}^t}) \\
\mathcal{IC}(\mathbb{1}_{C_u}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_u^*}) = \mathcal{IC}(\mathbb{1}_{C_x^t}) \\
\mathcal{IC}(\mathbb{1}_{C_x}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_x^*}) = \mathcal{IC}(\mathbb{1}_{C_u^t}) \\
\mathcal{IC}(\mathbb{1}_{C_{ux}}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_{ux}^*}) = \mathcal{IC}(\mathbb{1}_{C_0^t}) \\
\mathcal{IC}(\mathcal{L}_{C_x}) & \mapsto & \mathcal{IC}(\mathcal{L}_{C_0^*}) = \mathcal{IC}(\mathcal{L}_{C_{ux}^t}) \\
\mathcal{IC}(\mathcal{L}_{C_{ux}}) & \mapsto & \mathcal{IC}(\mathcal{L}_{C_u^*}) = \mathcal{IC}(\mathcal{L}_{C_x^t})
\end{array}$$

4.2.4. *Equivariant local systems on the regular conormal bundle.* The regular conormal bundle to the H-action on V decomposes into H-orbits

$$T^*_H(V)_{\mathrm{reg}} = T^*_{C_0}(V)_{\mathrm{reg}} \ \sqcup\ T^*_{C_u}(V)_{\mathrm{reg}} \ \sqcup\ T^*_{C_x}(V)_{\mathrm{reg}} \ \sqcup\ T^*_{C_{ux}}(V)_{\mathrm{reg}},$$

where each $T^*_C(V)_{\mathrm{reg}}$ is given below. In each case, the microlocal fundamental group A_C^{mic} is canonically identified with $Z(\widehat{G}) \cong \{\pm 1\}$.

C_0: Regular conormal bundle:

$$T^*_{C_0}(V)_{\mathrm{reg}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ u' & 0 & 0 & 0 \\ 0 & x' & 0 & 0 \\ 0 & 0 & u' & 0 \end{pmatrix} \ \middle|\ \begin{array}{c} u' \neq 0 \\ x' \neq 0 \end{array} \right\} = C_0 \times C_0^*$$

Base point:

$$(x_0, \xi_0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in T^*_{C_0}(V)_{\mathrm{reg}}$$

Fundamental groups:

$$1 = A_{x_0} \longleftarrow A_{(x_0,\xi_0)} \xrightarrow{\mathrm{id}} A_{\xi_0} = \{\pm 1\}$$

Local systems:

$\mathsf{Loc}_H(T^*_{C_0}(V)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_0}$	$\mathcal{L}_{\mathcal{O}_0}$
$\mathsf{Rep}(A_{(x_0,\xi_0)})$:	$+$	$-$

Pullback along the bundle map $T^*_{C_0}(V)_{\mathrm{sreg}} \to C_0$:

$$\begin{array}{ccc} \mathsf{Loc}_H(C_0) & \to & \mathsf{Loc}_H(T^*_{C_0}(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C_0} & \mapsto & \mathbb{1}_{\mathcal{O}_0} \\ & & \mathcal{L}_{\mathcal{O}_0} \end{array}$$

C_u: Regular conormal bundle:

$$T^*_{C_u}(V)_{\mathrm{reg}} = \left\{ \begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x' & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; \begin{array}{c} u \neq 0 \\ x' \neq 0 \end{array} \right\} = C_u \times C_u^*$$

Base point:

$$(x_1, \xi_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in T^*_{C_u}(V)_{\mathrm{reg}}$$

Fundamental groups:

$$1 = A_{x_1} \longleftarrow A_{(x_1,\xi_1)} \xrightarrow{\mathrm{id}} A_{\xi_1} = \{\pm 1\}$$

Local systems:

$\mathsf{Loc}_H(T^*_{C_u}(V)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_u}$	$\mathcal{L}_{\mathcal{O}_u}$
$\mathsf{Rep}(A_{(x_1,\xi_1)})$:	$+$	$-$

Pullback along the bundle map $T^*_{C_u}(V)_{\mathrm{sreg}} \to C_u$:

$$\begin{array}{ccc} \mathsf{Loc}_H(C_u) & \to & \mathsf{Loc}_H(T^*_{C_u}(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C_u} & \mapsto & \mathbb{1}_{\mathcal{O}_u} \\ & & \mathcal{L}_{\mathcal{O}_u} \end{array}$$

C_x: Regular conormal bundle:

$$T^*_{C_x}(V)_{\mathrm{reg}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ u' & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u' & 0 \end{pmatrix} \;\middle|\; \begin{array}{c} u' \neq 0 \\ x \neq 0 \end{array} \right\} = C_x \times C_x^*$$

Base point:

$$(x_2, \xi_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in T^*_{C_x}(V)_{\mathrm{reg}}.$$

Fundamental groups:

$$\{\pm 1\} = A_{x_2} \xleftarrow{\mathrm{id}} A_{(x_2,\xi_2)} \longrightarrow A_{\xi_2} = 1$$

Local systems:

$\mathsf{Loc}_H(T^*_{C_x}(V)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_x}$	$\mathcal{L}_{\mathcal{O}_x}$
$\mathsf{Rep}(A_{(x_2,\xi_2)})$:	$+$	$-$

Pullback along the bundle map $T^*_{C_x}(V)_{\mathrm{sreg}} \to C_x$:

$$\begin{array}{ccc} \mathsf{Loc}_H(C_x) & \to & \mathsf{Loc}_H(T^*_{C_x}(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C_x} & \mapsto & \mathbb{1}_{\mathcal{O}_x} \\ \mathcal{L}_{C_x} & \mapsto & \mathcal{L}_{\mathcal{O}_x} \end{array}$$

C_{ux}: Regular conormal bundle:

$$T^*_{C_{ux}}(V)_{\mathrm{reg}} = \left\{ \begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; \begin{array}{c} u \neq 0 \\ x \neq 0 \end{array} \right\} = C_{ux} \times C^*_{ux}$$

Base point:

$$(x_3, \xi_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in T^*_{C_{ux}}(V)_{\mathrm{reg}}$$

Fundamental groups:

$$\{\pm 1\} = A_{x_3} \xleftarrow{\mathrm{id}} A_{(x_3,\xi_3)} \longrightarrow A_{\xi_3} = 1$$

Local systems:

$\mathsf{Loc}_H(T^*_{C_{ux}}(V)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_{ux}}$	$\mathcal{L}_{\mathcal{O}_{ux}}$
$\mathsf{Rep}(A_{(x_3,\xi_3)})$:	$+$	$-$

Pullback along the bundle map $T^*_{C_{ux}}(V)_{\mathrm{sreg}} \to C_{ux}$:

$$\begin{array}{ccc} \mathsf{Loc}_H(C_{ux}) & \to & \mathsf{Loc}_H(T^*_{C_{ux}}(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C_{ux}} & \mapsto & \mathbb{1}_{\mathcal{O}_{ux}} \\ \mathcal{L}_{C_{ux}} & \mapsto & \mathcal{L}_{\mathcal{O}_{ux}} \end{array}$$

4.2.5. *Vanishing cycles of perverse sheaves.* The functor $\mathsf{Ev} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(T^*_H(V)_{\mathrm{reg}})$ is given on simple objects in Table 4.2.5.1

We now explain how to make these calculations.

(a) To compute $\mathsf{Ev}_{C_0}\mathcal{IC}(\mathbb{1}_{C_x})$ we look at the vanishing cycles

$$\mathsf{Ev}_{C_0}\mathcal{IC}(\mathbb{1}_{C_x}) = \mathsf{R}\Phi_{xx'}(\mathbb{1}_{\overline{C_x}\times C_0^*})|_{T^*_{C_0}(V)_{\mathrm{reg}}}[1].$$

The singular locus of xx' is $x = x' = 0$ but this is not part of $T^*_{C_0}(V)_{\mathrm{reg}}$, so $\mathsf{Ev}_{C_0}\mathcal{IC}(\mathbb{1}_{C_x}) = 0$. All the non-diagonal entries in the first four rows work similarly.

(b) To compute the last two rows of the tables above we consider the map $\pi : \widetilde{C}' \to C'$ which comes from taking a square root of x. Rather than directly applying Ev to $\mathcal{IC}(\mathcal{L}_{C'})$ we apply it to $\pi_!(\mathbb{1}_{\widetilde{C}'})$ and exploit the fact that we have already computed Ev for the $\mathcal{IC}$ sheaves of constant local systems. For example, in the case of $\mathsf{Ev}_{C_0}(\pi_!(\mathbb{1}_{\widetilde{C}_x}))$ we will compute:

$$\left(\pi_!\ \mathsf{R}\Phi_{x^2x'}(\mathbb{1}_{\widetilde{C}'} \boxtimes \mathbb{1}_{C_0^*})\right)_{T^*_{C_0}(V)_{\mathrm{reg}}}.$$

The singular locus is precisely $x = 0$ (noting that x' is not actually zero on the variety under consideration). The local structure of the singularity is that it is a smooth family (in the variable u') over the singularity of x^2x' over $\mathbb{A} \times \mathbb{G}_{\mathrm{m}}$ It

TABLE 4.2.5.1. ${}^p\mathsf{Ev} : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}})$ on simple objects, for $\lambda : W_F \to {}^LG$ given at the beginning of Section 4.

$$\begin{array}{ccc}
\mathsf{Per}_H(V) & \xrightarrow{{}^p\mathsf{Ev}} & \mathsf{Per}_H(T^*_H(V)_{\mathrm{reg}}) \\
\mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) \\
\mathcal{IC}(\mathbb{1}_{C_u}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_u}) \\
\mathcal{IC}(\mathbb{1}_{C_x}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_x}) \\
\mathcal{IC}(\mathbb{1}_{C_{ux}}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_{ux}}) \\
\mathcal{IC}(\mathcal{L}_{C_x}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_x}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}) \\
\mathcal{IC}(\mathcal{L}_{C_{ux}}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_{ux}}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_u})
\end{array}$$

$\mathcal{P}$	$\mathsf{Evs}_{C_0}\mathcal{P}$	$\mathsf{Evs}_{C_u}\mathcal{P}$	$\mathsf{Evs}_{C_x}\mathcal{P}$	$\mathsf{Evs}_{C_{ux}}\mathcal{P}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$+$	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_u})$	0	$+$	0	0
$\mathcal{IC}(\mathbb{1}_{C_x})$	0	0	$+$	0
$\mathcal{IC}(\mathbb{1}_{C_{ux}})$	0	0	0	$+$
$\mathcal{IC}(\mathcal{L}_{C_x})$	$-$	0	$-$	0
$\mathcal{IC}(\mathcal{L}_{C_{ux}})$	0	$-$	0	$-$

follows that the vanishing cycles on such a singularity is the sheaf supported on $x = 0$ associated to the non-trivial double cover $\sqrt{x'}$. Finally, by observing that the map π is an isomorphism on the support of $\mathsf{R}\Phi$, we conclude that:

$${}^p\mathsf{Ev}_{C_0}(\pi_!(\mathbb{1}_{\widetilde{C_x}})) = \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}).$$

The other entries are computed similarly.

4.2.6. *Normalization of Ev and the twisting local system.* From Table 4.2.5.1 we see that the twisting local system $\mathcal{T}$ is trivial in this case, so ${}^p\mathsf{NEv} = {}^p\mathsf{Ev}$.

4.2.7. *Fourier transform and vanishing cycles.* Compare the table below with the Fourier transform from Section 4.2.3 to confirm (21) in this example.

$$\begin{array}{ccccccc}
\mathsf{Per}_{H_\lambda}(V_\lambda) & \xrightarrow{{}^p\mathsf{Ev}} & \mathsf{Per}_H(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}}) & \xrightarrow{a_*} & \mathsf{Per}_H(T^*_{H_\lambda}(V^*_\lambda)_{\mathrm{reg}}) & \xleftarrow{\mathsf{Ev}^*} & \mathsf{Per}_{H_\lambda}(V^*_\lambda) \\
\mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}^*_0}) & \leftarrow\!\shortmid & \mathcal{IC}(\mathbb{1}_{C^*_0}) \\
\mathcal{IC}(\mathbb{1}_{C_u}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_u}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}^*_u}) & \leftarrow\!\shortmid & \mathcal{IC}(\mathbb{1}_{C^*_u}) \\
\mathcal{IC}(\mathbb{1}_{C_x}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_x}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}^*_x}) & \leftarrow\!\shortmid & \mathcal{IC}(\mathbb{1}_{C^*_x}) \\
\mathcal{IC}(\mathbb{1}_{C_{ux}}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_{ux}}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}^*_{ux}}) & \leftarrow\!\shortmid & \mathcal{IC}(\mathbb{1}_{C^*_{ux}}) \\
\mathcal{IC}(\mathcal{L}_{C_x}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_x}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}^*_x}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}^*_0}) & \leftarrow\!\shortmid & \mathcal{IC}(\mathcal{L}_{C^*_0}) \\
\mathcal{IC}(\mathcal{L}_{C_{ux}}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_{ux}}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_u}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}^*_{ux}}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}^*_u}) & \leftarrow\!\shortmid & \mathcal{IC}(\mathcal{L}_{C^*_u})
\end{array}$$

4.2.8. *Arthur sheaves.*

Arthur sheaf	pure L-packet sheaves	coronal perverse sheaves
$\mathcal{A}_{C_0}$	$\mathcal{IC}(\mathbb{1}_{C_0}) \oplus$	$\mathcal{IC}(\mathcal{L}_{C_x})$
$\mathcal{A}_{C_u}$	$\mathcal{IC}(\mathbb{1}_{C_u}) \oplus$	$\mathcal{IC}(\mathcal{L}_{C_{ux}})$
$\mathcal{A}_{C_x}$	$\mathcal{IC}(\mathbb{1}_{C_x}) \oplus \mathcal{IC}(\mathcal{L}_{C_x})$	
$\mathcal{A}_{C_{ux}}$	$\mathcal{IC}(\mathbb{1}_{C_{ux}}) \oplus \mathcal{IC}(\mathcal{L}_{C_{ux}})$	

4.3. **Adams-Barbasch-Vogan packets.**

4.3.1. *Admissible representations versus equivariant perverse sheaves.*

$\mathsf{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}}$	$\Pi_{\text{pure},\lambda}(G/F)$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$(\pi(\phi_0),0)$
$\mathcal{IC}(\mathbb{1}_{C_u})$	$(\pi(\phi_1),0)$
$\mathcal{IC}(\mathbb{1}_{C_x})$	$(\pi(\phi_2,+),0)$
$\mathcal{IC}(\mathbb{1}_{C_{ux}})$	$(\pi(\phi_3,+),0)$
$\mathcal{IC}(\mathcal{L}_{C_x})$	$(\pi(\phi_2,-),1)$
$\mathcal{IC}(\mathcal{L}_{C_{ux}})$	$(\pi(\phi_3,-),1)$

The Arthur parameters ψ_0 and ψ_3 correspond uniquely to the base points (x_0,ξ_0) and (x_3,ξ_3) from Section 4.2.4 under the map $Q_\lambda({}^LG) \to T^*_H(V)_{\text{reg}}$ given by [7, Theorem 4.1.1].

4.3.2. *ABV-packets.* Using Section 4.2.5 and the bijection of Section 4.3.1, we simply read off the ABV-packets:

$$\begin{array}{rcl}
\Pi^{\text{ABV}}_{\text{pure},\phi_0}(G/F) & = & \{[\pi(\phi_0),0],\ [\pi(\phi_2,-),1]\} \\
\Pi^{\text{ABV}}_{\text{pure},\phi_1}(G/F) & = & \{[\pi(\phi_1),0],\ [\pi(\phi_3,-),1]\} \\
\Pi^{\text{ABV}}_{\text{pure},\phi_2}(G/F) & = & \{[\pi(\phi_2,+),0],\ [\pi(\phi_2,-),1]\} \\
\Pi^{\text{ABV}}_{\text{pure},\phi_3}(G/F) & = & \{[\pi(\phi_3,+),0],\ [\pi(\phi_3,-),1]\}
\end{array}$$

Using Section 4.1.4, we see

$$\begin{array}{rcl}
\Pi_{\text{pure},\psi_0}(G/F) & = & \Pi^{\text{ABV}}_{\text{pure},\phi_0}(G/F) \\
\Pi_{\text{pure},\psi_3}(G/F) & = & \Pi^{\text{ABV}}_{\text{pure},\phi_3}(G/F),
\end{array}$$

thus verifying that Arthur packets are ABV-packets for admissible representations with infinitesimal parameter $\lambda : W_F \to {}^LG$ given at the beginning of Section 4.

4.3.3. *Stable distributions and endoscopy.* For $s \in Z(\widehat{G}) \cong \mu_2$, the virtual representations $\eta^{\mathsf{NEv}}_{\phi,s}$ of (29) are given by

$$\begin{array}{rcl}
\eta^{\mathsf{NEv}}_{\phi_0,1} & = & [\pi(\phi_0),0] + [\pi(\phi_2,-),1] \\
\eta^{\mathsf{NEv}}_{\phi_0,-1} & = & [\pi(\phi_0),0] - [\pi(\phi_2,-),1] \\
\eta^{\mathsf{NEv}}_{\phi_1,1} & = & [\pi(\phi_1),0] - [\pi(\phi_3,-),1] \\
\eta^{\mathsf{NEv}}_{\phi_1,-1} & = & [\pi(\phi_1),0] + [\pi(\phi_3,-),1] \\
\eta^{\mathsf{NEv}}_{\phi_2,1} & = & [\pi(\phi_2,+),0] + [\pi(\phi_2,-),1] \\
\eta^{\mathsf{NEv}}_{\phi_2,-1} & = & [\pi(\phi_2,+),0] - [\pi(\phi_2,-),1] \\
\eta^{\mathsf{NEv}}_{\phi_3,1} & = & [\pi(\phi_3,+),0] - [\pi(\phi_3,-),1] \\
\eta^{\mathsf{NEv}}_{\phi_3,-1} & = & [\pi(\phi_3,+),0] + [\pi(\phi_3,-),1]
\end{array}$$

Comparing with Section 4.1.6, this proves (28) in this example.

4.3.4. *Kazhdan-Lusztig conjecture.* Using the bijection of Section 4.3.1 we compare the normalised geometric multiplicity matrix from Section 4.2.2 with the multiplicity matrix

from Section 4.1.3:

$$m_{\mathrm{rep}} = \left(\begin{array}{cccc|cc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right), \qquad m'_{\mathrm{geo}} = \left(\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

Since ${}^t m_{\mathrm{rep}} = m'_{\mathrm{geo}}$, this confirms the Kazhdan-Lusztig conjecture as it applies to representations with infinitesimal parameter $\lambda : W_F \to {}^L G$ given at the beginning of Section 4.

4.3.5. *Aubert duality and Fourier transform.* To verify (30), use Vogan's bijection from Section 4.3.1 to compare Aubert duality from Section 4.1.5 with the Fourier transform from Section 4.2.3

To verify (31), observe that the twisting characters χ_ψ of A_ψ from Section 4.1.5 are trivial, as are the local systems $\mathcal{T}_\psi$ from Section 4.2.7.

4.3.6. *ABV-packets that are not pure Arthur packets.* The closed stratum C_0 and the open stratum C_3 are of Arthur type, while C_1 and C_2 are not of Arthur type. Thus, there are two ABV-packets that are not Arthur packets in this example:

$$\begin{array}{rcl} \Pi^{\mathrm{ABV}}_{\mathrm{pure},\phi_1}(G/F) & = & \{[\pi(\phi_1), 0],\ [\pi(\phi_3, -), 1]\} \\ \Pi^{\mathrm{ABV}}_{\mathrm{pure},\phi_2}(G/F) & = & \{[\pi(\phi_2, +), 0],\ [\pi(\phi_2, -), 1]\} \end{array}$$

From these we extract four stable distributions,

$$\begin{array}{rclrcl} \Theta^{G_0}_{\phi_1} & := & \operatorname{trace} \pi(\phi_1) & \Theta^{G_1}_{\phi_1} & := & \operatorname{trace} \pi(\phi_3, -) \\ \Theta^{G_0}_{\phi_2} & := & \operatorname{trace} \pi(\phi_2, +) & \Theta^{G_1}_{\phi_2} & := & -\operatorname{trace} \pi(\phi_2, -). \end{array}$$

We will see more interesting examples of ABV-packets that are not pure Arthur packets in Section 6.3.5.

4.4. **Endoscopy and equivariant restriction of perverse sheaves.** The material from Section 0.4 is trivial in this case.

5. SO(5) unipotent representations, singular infinitesimal parameter

In this example we encounter an L-packet of representations of $\mathrm{SO}(5, F)$ that is lifted from an L-packet of representations of $\mathrm{SO}(3, F) \times \mathrm{SO}(3, F)$. In Section 5.4.4 will see how this lifting may be understood through equivariant restriction of perverse sheaves on Vogan varieties, and their vanishing cycles.

Let $G = \mathrm{SO}(5)$. Then $H^1(F, G) \cong \mathbb{Z}/2\mathbb{Z}$. Let G_1 be the non-split form of G, as in Section 4. We consider admissible representations of $G(F)$ and $G_1(F)$ with infinitesimal parameter $\lambda : W_F \to {}^L G$ given by

$$\lambda(w) = \begin{pmatrix} |w|^{1/2} & 0 & 0 & 0 \\ 0 & |w|^{1/2} & 0 & 0 \\ 0 & 0 & |w|^{-1/2} & 0 \\ 0 & 0 & 0 & |w|^{-1/2} \end{pmatrix}.$$

5.1. **Arthur packets.**

5.1.1. *Parameters.* There are three Langlands parameters with infinitesimal parameter λ, up to $Z_{\widehat{G}}(\lambda)$-conjugacy, each of Arthur type. Set

$$\begin{array}{rcl} \psi_0(w,x,y) & := & \nu_2(y) \oplus \nu_2(y), \\ \psi_2(w,x,y) & := & \nu_2(x) \oplus \nu_2(y), \\ \psi_3(w,x,y) & := & \nu_2(x) \oplus \nu_2(x), \end{array}$$

and observe that ψ_0 and ψ_3 are Aubert dual while ψ_2 is self dual. Let ϕ_0, ϕ_2 and ϕ_3 be the associated Langlands parameters; thus,

$$\begin{array}{rcl} \phi_0(w,x) & := & \nu_2(d_w) \oplus \nu_2(d_w), \\ \phi_2(w,x) & := & \nu_2(x) \oplus \nu_2(d_w), \\ \phi_3(w,x) & := & \nu_2(x) \oplus \nu_2(x). \end{array}$$

5.1.2. *L-packets.* The pure component groups for these three Langlands parameters are

$$A_{\phi_0} = 1, \qquad A_{\phi_2} \cong \{\pm 1\}, \qquad A_{\phi_3} \cong \{\pm 1\}.$$

Thus, there are five admissible representations of two pure forms of SO(5) in play in this example. When arranged into L-packets, these representations are:

$$\begin{array}{rclrcl} \Pi_{\phi_0}(G_0(F)) & = & \{\pi(\phi_0)\}, & \Pi_{\phi_0}(G_1(F)) & = & \emptyset, \\ \Pi_{\phi_2}(G_0(F)) & = & \{\pi(\phi_2,+)\}, & \Pi_{\phi_2}(G_1(F)) & = & \{\pi(\phi_2,-)\}, \\ \Pi_{\phi_3}(G_0(F)) & = & \{\pi(\phi_3,+), \pi(\phi_3,-)\}, & \Pi_{\phi_3}(G_1(F)) & = & \emptyset. \end{array}$$

Of these five admissible representations, only $\pi(\phi_3,+)$ and $\pi(\phi_3,-)$ are tempered; these two representations are denoted by τ_2 and τ_1, respectively, in [18]. The admissible representation $\pi(\phi_0)$ is denoted by $L(\nu^{1/2}\zeta, \nu^{1/2}\zeta, 1)$ with $\zeta = 1$ in [18] and $\pi(\phi_2,+)$ is denoted by $L(\nu^{1/2}\zeta, \zeta\,\mathrm{St}_{\mathrm{SO}(3)})$ with $\zeta = 1$.

5.1.3. *Multiplicities in standard modules.* The standard module $M(\phi_0)$ is induced from the Levi subgroup $\mathrm{GL}(1,F) \times \mathrm{GL}(1,F) \times \mathrm{SO}(1,F)$ of $\mathrm{SO}(5,F)$; it is denoted by $\nu^{1/2}\zeta \times \nu^{1/2}\zeta \rtimes 1$ with $\zeta = 1$ in [18]. The standard module $M(\phi_2,+)$ is induced from the Levi subgroup $\mathrm{GL}(1,F) \times \mathrm{SO}(3,F)$ of $\mathrm{SO}(5,F)$; it is denoted by $\nu^{1/2}\zeta \rtimes \zeta\,\mathrm{St}_{\mathrm{SO}(3)}$ with $\zeta = 1$ in [18]. The standard module $M(\phi_3,\pm)$ coincides with the tempered representation $\pi(\phi_3,\pm)$. The 4×4 block in the following table may be deduced from [18, Proposition 3.3].

	$\pi(\phi_0)$	$\pi(\phi_2,+)$	$\pi(\phi_3,+)$	$\pi(\phi_3,-)$	$\pi(\phi_2,-)$
$M(\phi_0)$	1	1	1	1	0
$M(\phi_2,+)$	0	1	1	0	0
$M(\phi_3,+)$	0	0	1	0	0
$M(\phi_3,-)$	0	0	0	1	0
$M(\phi_2,-)$	0	0	0	0	1

5.1.4. *Arthur packets.* The component groups for the Arthur parameters in this example are

$$A_{\psi_0} \cong \{\pm 1\}, \qquad A_{\psi_2} \cong \{\pm 1\} \times \{\pm 1\}, \qquad A_{\psi_3} \cong \{\pm 1\}.$$

We may represent elements of each A_ψ as cosets with representatives taken from $\widehat{T}[2]$. The map $\widehat{T}[2] \to A_{\psi_0}$ is $s \mapsto s_1 s_2$; the map $\widehat{T}[2] \to A_{\psi_2}$ is $s \mapsto (s_1, s_2)$; the map $\widehat{T}[2] \to A_{\psi_3}$ is $s \mapsto s_1 s_2$.

The Arthur packets for Arthur parameters with infinitesimal parameter λ are:

$$\begin{array}{rclrcl}
\Pi_{\psi_0}(G_0(F)) & = & \{\pi(\phi_0), \pi(\phi_2,+)\}, & \Pi_{\psi_0}(G_1(F)) & = & \emptyset, \\
\Pi_{\psi_2}(G_0(F)) & = & \{\pi(\phi_2,+), \pi(\phi_3,-)\}, & \Pi_{\psi_2}(G_1(F)) & = & \{\pi(\phi_2,-)\}, \\
\Pi_{\psi_3}(G_0(F)) & = & \{\pi(\phi_3,+), \pi(\phi_3,-)\}, & \Pi_{\psi_3}(G_1(F)) & = & \emptyset.
\end{array}$$

We arrange these representations into pure Arthur packets in the table below.

pure Arthur packets	pure L-packet representations	coronal representations
$\Pi_{\mathrm{pure},\psi_0}(G/F)$	$[\pi(\phi_0), 0]$	$[\pi(\phi_2,+), 0]$
$\Pi_{\mathrm{pure},\psi_2}(G/F)$	$[\pi(\phi_2,+), 0]$, $[\pi(\phi_2,-), 1]$	$[\pi(\phi_3,-), 0]$
$\Pi_{\mathrm{pure},\psi_3}(G/F)$	$[\pi(\phi_3,+), 0]$, $[\pi(\phi_3,-), 0]$	

5.1.5. *Aubert duality.* Aubert duality for $G_0(F)$ and $G_1(F)$ are given by the following table.

π	$\hat{\pi}$
$\pi(\phi_0)$	$\pi(\phi_3,+)$
$\pi(\phi_2,+)$	$\pi(\phi_3,-)$
$\pi(\phi_3,+)$	$\pi(\phi_0,+)$
$\pi(\phi_3,-)$	$\pi(\phi_2,+)$
$\pi(\phi_2,-)$	$\pi(\phi_2,-)$

The twisting characters χ_{ψ_0} and χ_{ψ_3} are trivial. The twisting character χ_{ψ_2} of A_{ψ_2} is $\chi_{\psi_2}(s) = s_1 s_2 = \det(s)$. This is the first non-trivial twisting character to appear in this paper.

5.1.6. *Stable distributions and endoscopic transfer.* The stable distributions

$$\Theta_\psi^G = \sum_{\pi \in \Pi_\psi(G(F))} \langle s_\psi, \pi \rangle_\psi \operatorname{trace} \pi$$

attached the Arthur parameters are:

$$\begin{array}{rcl}
\Theta_{\psi_0}^G & = & \operatorname{trace}\pi(\phi_0) + \operatorname{trace}\pi(\phi_2,+) \\
\Theta_{\psi_2}^G & = & \operatorname{trace}\pi(\phi_2,+) - \operatorname{trace}\pi(\phi_3,-) \\
\Theta_{\psi_3}^G & = & \operatorname{trace}\pi(\phi_3,+) + \operatorname{trace}\pi(\phi_3,+).
\end{array}$$

The distributions

$$\Theta_{\psi,s}^G = \sum_{\pi \in \Pi_\psi(G_\delta(F))} \langle s s_\psi, \pi \rangle_\psi \operatorname{trace} \pi,$$

where $s \in Z_{\widehat{G}}(\psi)$, are obtained by transfer from endoscopic groups. The coefficients above are given by

$$\langle s s_\psi, \pi \rangle_\psi = \langle s_\psi, \pi \rangle_\psi \langle s, \pi \rangle_\psi$$

where $\langle s_\psi, \pi \rangle_\psi$ appear above while $\langle s, \pi \rangle_\psi$ is given by the tables below.

We now give $\langle \cdot \pi \rangle_\psi$ as a character of A_ψ, using the isomorphisms from Section 5.1.4.

π	$\langle \cdot, \pi \rangle_{\psi_0}$	$\langle \cdot, \pi \rangle_{\psi_2}$	$\langle \cdot, \pi \rangle_{\psi_3}$
$\pi(\phi_0)$	$+$	0	0
$\pi(\phi_2,+)$	$-$	$++$	0
$\pi(\phi_3,+)$	0	0	$+$
$\pi(\phi_3,-)$	0	$--$	$-$

Now we give the value of this character on the image of $s = \mathrm{diag}(s_1, s_2, s_2^{-1}, s_1^{-1}) \in \widehat{T}[2]$ in A_ψ.

π	$\langle s, \pi\rangle_{\psi_0}$	$\langle s, \pi\rangle_{\psi_2}$	$\langle s, \pi\rangle_{\psi_3}$
$\pi(\phi_0)$	1	0	0
$\pi(\phi_2, +)$	$s_1 s_2$	1	0
$\pi(\phi_3, +)$	0	0	1
$\pi(\phi_3, -)$	0	$s_1 s_2$	$s_1 s_2$

For instance, if we take $s = \mathrm{diag}(1, -1, -1, 1) \in \widehat{T}[2]$ then

$$\begin{aligned}
\Theta^G_{\psi_0,s} &= \mathrm{trace}\,\pi(\phi_0) - \mathrm{trace}\,\pi(\phi_2, +),\\
\Theta^G_{\psi_2,s} &= \mathrm{trace}\,\pi(\phi_2, +) + \mathrm{trace}\,\pi(\phi_3, -),\\
\Theta^G_{\psi_3,s} &= \mathrm{trace}\,\pi(\phi_3, +) - \mathrm{trace}\,\pi(\phi_3, -).
\end{aligned}$$

In this case, the elliptic endoscopic group G' for G determined by s is $G' = \mathrm{SO}(3) \times \mathrm{SO}(3)$, split over F.

5.2. **Vanishing cycles of perverse sheaves.** We now assemble the geometric tools needed to calculate the Arthur packets, stable distributions and endoscopic transfer described above.

5.2.1. *Vogan variety and its conormal bundle.*

$$V = \left\{ \left(\begin{array}{cc|cc} 0 & 0 & z & x \\ 0 & 0 & y & -z \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \mid x, y, z \right\} \qquad V^* = \left\{ \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline z' & y' & 0 & 0 \\ x' & -z' & 0 & 0 \end{array} \right) \mid x', y', z' \right\}$$

so

$$T^*(V) = \left\{ \left(\begin{array}{cc|cc} 0 & 0 & z & x \\ 0 & 0 & y & -z \\ \hline z' & y' & 0 & 0 \\ x' & -z' & 0 & 0 \end{array} \right) \mid \begin{array}{c} x, y, z \\ x', y', z' \end{array} \right\} \subset \mathrm{sp}(4)$$

The cotangent bundle $T^*(V)$ comes equipped with an action of

$$H := Z_{\widehat{G}}(\lambda) = \left\{ \left(\begin{array}{cc|cc} a_1 & b_1 & 0 & 0 \\ c_1 & d_1 & 0 & 0 \\ \hline 0 & 0 & a_2 & b_2 \\ 0 & 0 & c_2 & d_2 \end{array} \right) \in \mathrm{Sp}(4) \right\}.$$

We will write $h_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ and $h_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$. Then $h_2 = h_1 \det h_1^{-1}$, by the choice of symplectic form J in Section 4. In particular, $H \cong \mathrm{GL}(2)$. The action of H on V, V^* and $T^*(V)$ is given by

$$\begin{aligned}
h \cdot \begin{pmatrix} z & x \\ y & -z \end{pmatrix} &= h_1 \begin{pmatrix} z & x \\ y & -z \end{pmatrix} h_2^{-1}\\
h \cdot \begin{pmatrix} z' & y' \\ x' & -z' \end{pmatrix} &= h_2 \begin{pmatrix} z' & y' \\ x' & -z' \end{pmatrix} h_1^{-1}.
\end{aligned}$$

The conormal bundle is

$$T^*_H(V) = \left\{ \left(\begin{array}{cc|cc} & & z & x \\ & & y & -z \\ \hline z' & y' & & \\ x' & -z' & & \end{array} \right) \;\middle|\; \begin{array}{c} zz' + xx' = 0 \\ zz' + yy' = 0 \\ zx' - yz' = 0 \\ xz' - zy' = 0 \end{array} \right\}$$

5.2.2. *Equivariant local systems.*

C_0: Regular conormal bundle above the closed H-orbit $C_0 \subset V$:

$$T^*_{C_0}(V)_{\mathrm{reg}} = \left\{ \left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 0 & 0 \\ \hline z' & y' & & \\ x' & -z' & & \end{array} \right) \;\middle|\; x'y' - z'^2 \neq 0 \right\}$$

Base point:

$$(x_0, \xi_0) = \left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 1 & & \\ 1 & 0 & & \end{array} \right) \in T^*_{C_0}(V)_{\mathrm{reg}}$$

Fundamental groups:

$$\begin{array}{ccccc} & & \widehat{T}[2] & & \\ & & \Big\downarrow{\scriptstyle s \mapsto s_1 s_2} & & \\ 1 = A_{x_0} & \longleftarrow & A_{(x_0,\xi_0)} & \xrightarrow{\mathrm{id}} & A_{\xi_0} \cong \{\pm 1\} \end{array}$$

Local systems:

$\mathsf{Loc}_H(T^*_{C_0}(V)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_0}$	$\mathcal{L}_{\mathcal{O}_0}$
$\mathsf{Rep}(A_{(x_0,\xi_0)})$:	$+$	$-$

Pullback along the bundle map $T^*_{C_0}(V)_{\mathrm{sreg}} \to C_0$:

$$\begin{array}{ccc} \mathsf{Loc}_H(C_0) & \to & \mathsf{Loc}_H(T^*_{C_0}(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C_0} & \mapsto & \mathbb{1}_{\mathcal{O}_0} \\ & & \mathcal{L}_{\mathcal{O}_0} \end{array}$$

C_2: Regular conormal bundle above $C_2 \subset V$:

$$T^*_{C_2}(V)_{\mathrm{reg}} = \left\{ \left(\begin{array}{cc|cc} & & z & x \\ & & y & -z \\ \hline z' & y' & & \\ x' & -z' & & \end{array} \right) \;\middle|\; \begin{array}{c} xy + z^2 = 0 \\ [x:y:z] = [y':x':z'] \end{array} \right\}$$

Base point:

$$(x_2, \xi_2) = \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 0 & 0 \\ \hline 0 & 1 & & \\ 0 & 0 & & \end{array} \right) \in T^*_{C_2}(V)_{\mathrm{reg}}$$

Fundamental groups:

$$\begin{tikzcd}
 & \widehat{T}[2] \arrow[d, "s\mapsto(s_1,s_2)", "\cong"'] & \\
\{\pm 1\} = A_{x_2} & A_{(x_2,\xi_2)} \arrow[l, "s_1\leftarrow\!\shortmid(s_1,s_2)"'] \arrow[r, "(s_1,s_2)\mapsto s_2"] & A_{\xi_2} = \{\pm 1\}
\end{tikzcd}$$

Local systems:

$\mathsf{Loc}_H(T^*_{C_2}(V)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_2}$	$\mathcal{L}_{\mathcal{O}_2}$	$\mathcal{F}_{\mathcal{O}_2}$	$\mathcal{E}_{\mathcal{O}_2}$
$\mathsf{Rep}(A_{(x_2,\xi_2)})$:	$++$	$--$	$-+$	$+-$

Pullback along the bundle map $T^*_{C_2}(V)_{\mathrm{sreg}} \to C_2$:

$$\begin{array}{ccc}
\mathsf{Loc}_H(C_2) & \to & \mathsf{Loc}_H(T^*_{C_2}(V)_{\mathrm{sreg}}) \\
\mathbb{1}_{C_2} & \mapsto & \mathbb{1}_{\mathcal{O}_2} \\
 & & \mathcal{L}_{\mathcal{O}_2} \\
\mathcal{F}_{C_2} & \mapsto & \mathcal{F}_{\mathcal{O}_2} \\
 & & \mathcal{E}_{\mathcal{O}_2}
\end{array}$$

C_3: Regular conormal bundle above $C_3 \subset V$:

$$T^*_{C_3}(V)_{\mathrm{reg}} = \left\{ \left(\begin{array}{cc|cc} & & z & x \\ & & y & -z \\ \hline 0 & 0 & & \\ 0 & 0 & & \end{array} \right) \;\middle|\; xy + z^2 \neq 0 \right\}$$

Base point:

$$(x_3, \xi_3) = \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 1 & 0 \\ \hline 0 & 0 & & \\ 0 & 0 & & \end{array} \right) \in T^*_{C_3}(V)_{\mathrm{reg}}$$

Fundamental groups:

$$\begin{tikzcd}
 & \widehat{T}[2] \arrow[d, "s_1 s_2\leftarrow\!\shortmid s"'] & \\
\{\pm 1\} \cong A_{x_3} & A_{(x_3,\xi_3)} \arrow[l, "\mathrm{id}"'] \arrow[r] & A_{\xi_3} = 1
\end{tikzcd}$$

Local systems:

$\mathsf{Loc}_H(T^*_{C_3}(V)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_3}$	$\mathcal{L}_{\mathcal{O}_3}$
$\mathsf{Rep}(A_{(x_3,\xi_3)})$:	$+$	$-$

Pullback along the bundle map $T^*_{C_3}(V)_{\mathrm{sreg}} \to C_3$:

$$\begin{array}{ccc}
\mathsf{Loc}_H(C_3) & \to & \mathsf{Loc}_H(T^*_{C_3}(V)_{\mathrm{sreg}}) \\
\mathbb{1}_{C_3} & \mapsto & \mathbb{1}_{\mathcal{O}_3} \\
\mathcal{L}_{C_3} & \mapsto & \mathcal{L}_{\mathcal{O}_3}
\end{array}$$

5.2.3. *Equivariant perverse sheaves.* The following table is helpful to understand the simple objects in $\mathsf{Per}_H(V)$.

$\mathcal{P}$	$\mathcal{P}\|_{C_0}$	$\mathcal{P}\|_{C_2}$	$\mathcal{P}\|_{C_3}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathbb{1}_{C_0}[0]$	0	0
$\mathcal{IC}(\mathbb{1}_{C_2})$	$\mathbb{1}_{C_0}[2]$	$\mathbb{1}_{C_2}[2]$	0
$\mathcal{IC}(\mathbb{1}_{C_3})$	$\mathbb{1}_{C_0}[3]$	$\mathbb{1}_{C_2}[3]$	$\mathbb{1}_{C_3}[3]$
$\mathcal{IC}(\mathcal{L}_{C_3})$	$\mathbb{1}_{C_0}[1]$	0	$\mathcal{L}_{C_3}[3]$
$\mathcal{IC}(\mathcal{F}_{C_2})$	0	$\mathcal{F}_{C_2}[2]$	0

We now explain how we made these calculations:

(a) The first and third row of these tables are computed using the observation that when $\overline{C}$ is smooth, the sheaf $\mathbb{1}_{\overline{C}}[\dim(C)]$ is perverse.

(b) For the second row, the relevant cover $\widetilde{C}_2^{(1)}$ is the blowup of the nilcone at the origin. We readily find using the decomposition theorem for semi-small maps that

$$\pi_2^{(1)}{}_!(\mathbb{1}_{\widetilde{C}_2^{(1)}}[2]) = \mathcal{IC}(\mathbb{1}_{C_2}) \oplus \mathcal{IC}(\mathbb{1}_{C_0}).$$

Proper base change and exactness allows us to deduce the fibres of $\mathcal{IC}(\mathbb{1}_{C_2})$ using what we already know about $\mathcal{IC}(\mathbb{1}_{C_0})$.

(c) For the fourth row, we consider the double cover which arises from taking the square root of the determinant. Although this is singular at the origin, blowing up resolves this singularity. An alternate model for this blowup is the cover:

$$\widetilde{C}_3 = \left\{[a:b], (x,y,z) \in \mathbb{P}^1 \times V \ \mid \ -a^2x + 2abz + b^2y = 0\right\}$$

with the obvious map π_3 to V. The decomposition theorem for semi-small maps yields

$$\pi_{3!}(\mathbb{1}_{\widetilde{C}_3}[3]) = \mathcal{IC}(\mathcal{L}_{C_3}) \oplus \mathcal{IC}(\mathbb{1}_{C_3}).$$

Proper base change and exactness again allows us to deduce the entries for $\mathcal{IC}(\mathcal{L}_{C_3})$, the key observation being that the map is $2:1$ over C_3, an isomorphism over C_2 and the fibre over C_0 is $\mathbb{P}^1$.

(d) Finally the fifth row is computed by considering the "symmetric squares" cover of the nilcone given by $\pi_2^{(2)} : (a,b) \mapsto (-a^2, b^2, ab)$. This map is $2:1$ over C_2 and an isomorphism over C_0; we readily confirm using the decomposition theorem for finite maps that

$$\pi_2^{(2)}{}_!(\mathbb{1}_{\tilde{C}}[2]) = \mathcal{IC}(\mathcal{F}_{C_2}) \oplus \mathcal{IC}(\mathbb{1}_{C_2}).$$

Computing the entries in the table is now immediate using our understanding of the fibres and what we already know about $\mathcal{IC}(\mathbb{1}_{C_2})$.

From this, we easily find the normalised geometric multiplicity matrix.

	$\mathbb{1}^\natural_{C_0}$	$\mathbb{1}^\natural_{C_2}$	$\mathbb{1}^\natural_{C_3}$	$\mathcal{L}^\natural_{C_3}$	$\mathcal{F}^\natural_{C_2}$
$\mathbb{1}^\sharp_{C_0}$	1	0	0	0	0
$\mathbb{1}^\sharp_{C_2}$	1	1	0	0	0
$\mathbb{1}^\sharp_{C_3}$	1	1	1	0	0
$\mathcal{L}^\sharp_{C_3}$	1	0	0	1	0
$\mathcal{F}^\sharp_{C_2}$	0	0	0	0	1

TABLE 5.2.5.1. $\mathsf{Ev} : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}})$ on simple objects, for $\lambda : W_F \to {}^L G$ given at the beginning of Section 5.

$$\begin{array}{ccc}
\mathsf{Per}_{H_\lambda}(V_\lambda) & \xrightarrow{{}^p\mathsf{Ev}} & \mathsf{Per}_H(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}}) \\
\mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) \\
\mathcal{IC}(\mathbb{1}_{C_2}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}) \\
\mathcal{IC}(\mathbb{1}_{C_3}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_3}) \\
\mathcal{IC}(\mathcal{L}_{C_3}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathbb{1}_{\mathcal{O}_2}) \\
\mathcal{IC}(\mathcal{F}_{C_2}) & \mapsto & \mathcal{IC}(\mathcal{E}_{\mathcal{O}_2})
\end{array}$$

$\mathcal{P}$	$\mathsf{Evs}_{\psi_0}\mathcal{P}$	$\mathsf{Evs}_{\psi_2}\mathcal{P}$	$\mathsf{Evs}_{\psi_3}\mathcal{P}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$+$	0	0
$\mathcal{IC}(\mathbb{1}_{C_2})$	$-$	$--$	0
$\mathcal{IC}(\mathbb{1}_{C_3})$	0	0	$+$
$\mathcal{IC}(\mathcal{L}_{C_3})$	0	$++$	$-$
$\mathcal{IC}(\mathcal{F}_{C_2})$	0	$+-$	0

5.2.4. *Cuspidal support decomposition and Fourier transform.* Cuspidal Levi subgroups for $\widehat{G}$ were given in Section 4.2.3, so the cuspidal support decomposition of $\mathsf{Per}_{H_\lambda}(V_\lambda)$ takes the same form here:

$$\mathsf{Per}_{H_\lambda}(V_\lambda) = \mathsf{Per}_{H_\lambda}(V_\lambda)_{\widehat{T}} \oplus \mathsf{Per}_{H_\lambda}(V_\lambda)_{\widehat{M}}.$$

However, simple objects in these two subcategories are quite different in this case:

$\mathsf{Per}_H(V)_{\tilde{T}}$	$\mathsf{Per}_H(V)_{\tilde{M}}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	
$\mathcal{IC}(\mathbb{1}_{C_2})$	$\mathcal{IC}(\mathcal{F}_{C_2})$
$\mathcal{IC}(\mathbb{1}_{C_3})$	
$\mathcal{IC}(\mathcal{L}_{C_3})$	

Here we record the functor $\mathsf{Ft} : \mathsf{Per}_H(V) \to \mathsf{Per}_H(V^*)$ on simple objects, and the composition of that functor with the equivalence $\mathsf{Per}_H(V^*) \to \mathsf{Per}_H(V)$ described in Section 0.3.4; the composition is the functor ${}^\wedge : \mathsf{Per}_H(V) \to \mathsf{Per}_H(V)$ also discussed in Section 0.3.4.

$$\begin{array}{ccccc}
\mathsf{Per}_H(V) & \xrightarrow{\mathsf{Ft}} & \mathsf{Per}_H(V^*) & \longrightarrow & \mathsf{Per}_H(V) \\
\mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_0^*}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_3}) \\
\mathcal{IC}(\mathbb{1}_{C_2}) & \mapsto & \mathcal{IC}(\mathcal{L}_{C_0^*}) & \mapsto & \mathcal{IC}(\mathcal{L}_{C_3}) \\
\mathcal{IC}(\mathbb{1}_{C_3}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_3^*}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_0}) \\
\mathcal{IC}(\mathcal{L}_{C_3}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_2^*}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_2}) \\
\mathcal{IC}(\mathcal{F}_{C_2}) & \mapsto & \mathcal{IC}(\mathcal{F}_{C_2^*}) & \mapsto & \mathcal{IC}(\mathcal{F}_{C_2})
\end{array}$$

Note that the Fourier transform respects the cuspidal support decomposition.

5.2.5. *Vanishing cycles.* Table 5.2.5.1 presents the calculation of Ev_C on simple objects.

Rows 1 and 3 are straightforward, arguing as in Section 4.2.5, (a). Here we show how to compute rows 2, 4 and 5.

(a) To compute $\mathsf{Ev}_{C_2}\mathcal{IC}(\mathbb{1}_{C_2})$ we first consider

$$\mathsf{R}\Phi_{xx'+yy'+2zz'}(\mathbb{1}_{\overline{C_2}\times C_2^*})$$

We have the equations $xy + z^2 = 0$ and $x'y' + z'^2 = 0$. We localize away from $y' = 0$ and $x = 0$ and we can rewrite this as:

$$\mathsf{R}\Phi_{x\frac{z'^2}{y'}+y'\frac{z^2}{x}+2zz'}(\mathbb{1})$$

where we are working on the the subvariety of $\mathbb{A}^4$ with coordinates x, y', z, z' where x, y' are not zero. We rewrite this as:

$$\mathsf{R}\Phi_{\frac{1}{xy'}(zy'+xz')^2}(\mathbb{1})$$

But this is just the sheaf on $zy' + xz' = 0$ associated to the double cover with $\sqrt{xy'}$.

Note that:

$$\begin{pmatrix} -x & z \\ z & -\frac{z^2}{x} \end{pmatrix} \begin{pmatrix} \frac{z'^2}{y'} & z' \\ z' & y' \end{pmatrix} = \begin{pmatrix} -x\frac{z'^2}{y'} + zz' & -xz' + zy' \\ z\frac{z'^2}{y'} - z'\frac{z^2}{x} & zz' - y'\frac{z^2}{x} \end{pmatrix}$$

and so the equation $zy' + xz' = 0$ implies this product is zero. This says the support is the regular part of the conormal bundle.

Note that points where $y' \neq 0$ and $x = 0$ are smooth, so if we really wanted to compute the entire sheaf we only need to consider the additional local chart where we localize away from from $x' = 0$ and $y = 0$.

In this case we symmetrically obtain the sheaf on $zx' + yz' = 0$ associated to the double cover with $\sqrt{yx'}$.

Indeed, this cover is the cover:

$$\mathrm{Spec}(k[a^2, b^2, ab, a'^2, b'^2, a'b', ab', aa', ba', bb'])$$

(that is the subvariety of $\mathbb{A}^{10}$ with all of the implied relations) where the covering is associated to

$$x \to a^2, y \to b^2, z \to ab, x' \to a'^2, y' \to b'^2, z' \to a'b'$$

(this is the diagonal quotient of the product of the symmetric squares).

After smooth pullback to this cover we are computing:

$$\mathsf{R}\Phi_{a^2a'^2+b^2b'^2+2aa'bb'}(\mathbb{1}) = \mathsf{R}\Phi_{(aa'+bb')^2}(\mathbb{1})$$

is trivial. From which it follows the original sheaf does indeed trivialize over this double cover.

We now see what happens when we do things the ABV way:

We pick a point $x = 1, y = 0, z = 0$, we pick a function whose derivative is regular y.

$$\mathsf{R}\Phi_y \mathbb{1}_{(1,0,0)}$$

But near the point $x = 1$ we have $x \neq 0$ and so $y = -\frac{z^2}{x}$ and so we are computing:

$$\mathsf{R}\Phi_{-\frac{1}{x}z^2}\mathbb{1}_{(1,0,0)}$$

This is the sheaf associated to taking the square root of x at the point $z = 0$.

(b) In order to compute $\mathsf{Ev}_{C_2}(\mathcal{IC}(\mathcal{F}_2))$ we consider

$$\mathsf{R}\Phi_{a^2x'+b^2y'+2abz'}(\mathbb{1})$$

on the cover

$$\mathrm{Spec}(k[a^2, b^2, ab, x', y', z']/(x'y' + z'^2))$$

I claim this sheaf is associated to the double cover:

$$\mathrm{Spec}(k[a^2, b^2, ab, a^2, b^2, ab])$$

By smooth pullback it certainly trivializes oover this cover, to see that it is non-trival.

To see that what it is, we can localize away from either $x' = 0$ or $y' = 0$, we discuss the former case we obtain:

$$\mathsf{R}\Phi_{\frac{1}{y'}(az'+by')^2}(\mathbb{1})$$

Which is the sheaf associated to the cover $\sqrt{y'}$ over the zero locus of $az' + by'$.

This shows that $\mathsf{Ev}_{C_2}(\mathcal{IC}(\mathcal{F}_2))$ is not the pullback of $\mathcal{F}_2$, but rather, twist of that by the same twist above, or equivalently, the pullback on the transpose side:

(c) To compute $\mathsf{Ev}_{C_0}\mathcal{IC}(\mathbb{1}_{C_2})$ we first consider

$$\mathsf{R}\Phi_{xx'+yy'+2zz'}(\mathbb{1}_{\widetilde{C}_2^{(1)}\times C_0^*})$$

with

$$\widetilde{C}_2^{(1)} \times C_0^* = \{[a:b], (x,y,z), (x',y',z') \ \mid \ -ax + bz = 0,\ az + by = 0\}.$$

The Jacobian condition for smoothness tells us that this is singular when $x = y = z = 0$ and

$$-x'\frac{b}{a} + y'\frac{a}{b} = 2z';$$

re-homogenizing gives:

$$-a^2x' + 2abz' + b^2y' = 0.$$

The restriction of $\widetilde{C}_2^{(1)} \times C_0^* \to C_2 \times C_0^*$ to the singular locus to give the non-trivial double cover of C_0^*. From this we conclude

$$\mathsf{Ev}_{C_0}\pi_2^{(1)}!\mathbb{1}_{\widetilde{C}_2^{(1)}}[2] = \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}) \oplus \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}).$$

As we already know that $\mathcal{IC}(\mathbb{1}_{C_0})$ is the source of the second term, we conclude that

$$\mathsf{Ev}_{C_0}\mathcal{IC}(\mathbb{1}_{C_2}) = \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}).$$

(d) To compute $\mathsf{Ev}_{C_0}\mathcal{IC}(\mathcal{F}_{C_2})$ we consider

$$\mathsf{R}\Phi_{-a^2x'+2abz'+b^2y'}(\mathbb{1}_{\widetilde{C}_2^{(2)}\times C_0^*}).$$

By passing to local charts we can describe the detailed local structure of the singularity, using that we know at least one of x', y', z' is not zero. However, in this case, the key observation we need is that $\pi_2^{(2)}$ is an isomorphism above the singular locus and that we will obtain a rank-1 sheaf on the singular locus. The former condition is easily checked using the Jacobian condition, the latter can be checked by restricting to the chart where none of x', y', z' are zero, and taking an appropriate coordinate change. We already know this rank-1 sheaf is explained by $\mathsf{Ev}_{C_0}\mathcal{IC}(\mathbb{1}_{C_2})$, and hence

$$\mathsf{Ev}_{C_0}\mathcal{IC}(\mathcal{F}_{C_2}) = 0.$$

(e) The smoothness of $\overline{C_3} = V$ makes the computation of $\mathsf{Ev}\,\mathcal{IC}(\mathbb{1}_{C_3})$ straightforward. Indeed for $\mathsf{Ev}_{C_0}(\mathcal{IC}(\mathbb{1}_{C_3}))$ we consider

$$\mathsf{R}\Phi_{xx'+2zz'+yy'}(\mathbb{1}_{C_3\times C_0^*}).$$

But the singular locus is $x = x' = y = y' = z = z' = 0$, which is not a point in C_0^*. So we obtain the zero sheaf on $\mathcal{O}_{C_2}$

For $\mathsf{Ev}_{C_2}(\mathcal{IC}(\mathbb{1}_{C_3}))$ we consider

$$\mathsf{R}\Phi_{xx'+2zz'+yy'}(\mathbb{1}_{C_3\times C_2^*}).$$

Where we have the equation ${z'}^2 + x'y' = 0$ But the singular locus is $x = x' = y = y' = z = z' = 0$, which is not a point in C_2^*. So we obtain the zero sheaf on $\mathcal{O}_{C_2}$

Finally for $\mathsf{Ev}_{C_3}(\mathcal{IC}(\mathbb{1}_{C_3}))$ we consider

$$\mathsf{R}\Phi_0(\mathbb{1}_{C_3\times C_3^*}).$$

So we obtain the constant sheaf on $\mathcal{O}_{C_3}$

(f) To compute $\mathsf{Ev}_{C_0}\,\mathcal{IC}(\mathcal{L}_{C_3})$ we are considering

$$\mathsf{R}\Phi_{xx'+2zz'+yy'}(\mathbb{1}_{\widetilde{C_3}\times C_0^*}).$$

on the cover

$$\left\{[a:b],(x,y,z),(x',y',z') \;\mid\; -a^2x + 2abz + b^2y = 0\right\}.$$

By computing the Jacobian we find the singular locus avoids the regular part of the conormal bundle.

(g) To compute $\mathsf{Ev}_{C_2}\,\mathcal{IC}(\mathcal{L}_{C_3})$ we are considering

$$\mathsf{R}\Phi_{xx'+2zz'+yy'}(\mathbb{1}_{\widetilde{C_3}\times C_2^*}).$$

on the cover

$$\left\{[a:b],(x,y,z),(x',y',z') \;\mid\; -a^2x + 2abz + b^2y = 0, {z'}^2 + x'y' = 0\right\}.$$

However we may localize away from the exceptional divisor and instead work with the cover:

$$\left\{(d),(x,y,z),(x',y',z') \;\mid\; d^2 = z^2 + xy, {z'}^2 + x'y' = 0\right\}.$$

We further localize away from $y' = 0$ and $x = 0$ so we may rewrite this as:

$$\mathsf{R}\Phi_{\frac{-1}{xy'}(x^2{z'}^2 - 2xy'zz' + (z^2-d^2){y'}^2)}(\mathbb{1}_{\widetilde{C_3}\times C_2^*}).$$

but this is:

$$\mathsf{R}\Phi_{\frac{-1}{xy'}((xz'-zy')^2 - d^2{y'}^2)}(\mathbb{1}_{\widetilde{C_3}\times C_2^*}).$$

which is in turn

$$\mathsf{R}\Phi_{\frac{-1}{xy'}(xz'-zy'-dy')(xz'-zy'+dy')}(\mathbb{1}_{\widetilde{C_3}\times C_2^*}).$$

The functions $(xz' - zy' - dy')$ and $(xz' - zy' + dy')$ being smooth, and xy' being non-zero gives that the vanishing cycles are the constant sheaf on the intersection of their zero loci, which is precisely $\mathcal{O}_{C_2}$.

The proper pushforward is an isomorphism on the regular conormal vectors, and so we obtain the constant sheaf on $\mathcal{O}_{C_2}$.

TABLE 5.2.6.1. $\mathsf{NEv} : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}})$ on simple objects, for $\lambda : W_F \to {}^L G$ given at the beginning of Section 5.

$$\begin{array}{ccc}
\mathsf{Per}_{H_\lambda}(V_\lambda) & \xrightarrow{{}^p\mathsf{NEv}} & \mathsf{Per}_H(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}}) \\
\mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) \\
\mathcal{IC}(\mathbb{1}_{C_2}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}) \\
\mathcal{IC}(\mathbb{1}_{C_3}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_3}) \\
\mathcal{IC}(\mathcal{L}_{C_3}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2}) \\
\mathcal{IC}(\mathcal{F}_{C_2}) & \mapsto & \mathcal{IC}(\mathcal{F}_{\mathcal{O}_2})
\end{array}$$

$\mathcal{P}$	$\mathsf{NEvs}_{\psi_0}\mathcal{P}$	$\mathsf{NEvs}_{\psi_2}\mathcal{P}$	$\mathsf{NEvs}_{\psi_3}\mathcal{P}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$+$	0	0
$\mathcal{IC}(\mathbb{1}_{C_2})$	$-$	$++$	0
$\mathcal{IC}(\mathbb{1}_{C_3})$	0	0	$+$
$\mathcal{IC}(\mathcal{L}_{C_3})$	0	$--$	$-$
$\mathcal{IC}(\mathcal{F}_{C_2})$	0	$-+$	0

(h) To compute $\mathsf{Ev}_{C_3}\mathcal{IC}(\mathcal{L}_{C_3})$ we are considering

$$\mathsf{R}\Phi_0(\mathbb{1}_{\widetilde{C_3}\times C_3^*}).$$

We can thus conclude that we obtain $\mathcal{L}_{O_3}$

5.2.6. *Normalization of Ev and the twisting local system.* Using Table 5.2.5.1 we find our first case when the equivariant local system $\mathcal{T}$ is non-trivial.

	ψ_0	ψ_2	ψ_3
$\mathcal{T}_\psi$	$+$	$--$	$+$

We use $\mathcal{T}$ in Table 5.2.6.1 to calculate ${}^p\mathsf{NEv} : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}})$ in two forms; compare with Table 5.2.5.1

5.2.7. *Fourier transform and vanishing cycles.* We may now verify (21) by comparing the functors below with the Fourier transform appearing in Section 5.2.4.

$$\begin{array}{ccccccc}
\mathsf{Per}_{H_\lambda}(V_\lambda) & \xrightarrow{{}^p\mathsf{NEv}} & \mathsf{Per}_H(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}}) & \xrightarrow{a_*} & \mathsf{Per}_H(T^*_{H_\lambda}(V^*_\lambda)_{\mathrm{reg}}) & \xleftarrow{{}^p\mathsf{Ev}^*} & \mathsf{Per}_{H_\lambda}(V^*_\lambda) \\
\mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0^*}) & \leftarrow\!\shortmid & \mathcal{IC}(\mathbb{1}_{C_0^*}) \\
\mathcal{IC}(\mathbb{1}_{C_2}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_2^*}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0^*}) & \leftarrow\!\shortmid & \mathcal{IC}(\mathcal{L}_{C_0^*}) \\
\mathcal{IC}(\mathbb{1}_{C_3}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_3}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_3^*}) & \leftarrow\!\shortmid & \mathcal{IC}(\mathbb{1}_{C_3^*}) \\
\mathcal{IC}(\mathcal{L}_{C_3}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_3^*}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2^*}) & \leftarrow\!\shortmid & \mathcal{IC}(\mathbb{1}_{C_2^*}) \\
\mathcal{IC}(\mathcal{F}_{C_2}) & \mapsto & \mathcal{IC}(\mathcal{F}_{\mathcal{O}_2}) & \mapsto & \mathcal{IC}(\mathcal{F}_{\mathcal{O}_2^*}) & \leftarrow\!\shortmid & \mathcal{IC}(\mathcal{F}_{C_2^*})
\end{array}$$

5.2.8. *Arthur sheaves.*

Arthur sheaf	packet sheaves	coronal sheaves
$\mathcal{A}_{C_0}$	$\mathcal{IC}(\mathbb{1}_{C_0}) \oplus$	$\mathcal{IC}(\mathbb{1}_{C_2})$
$\mathcal{A}_{C_2}$	$\mathcal{IC}(\mathbb{1}_{C_2}) \oplus \mathcal{IC}(\mathcal{F}_{C_2}) \oplus$	$\mathcal{IC}(\mathcal{L}_{C_3})$
$\mathcal{A}_{C_3}$	$\mathcal{IC}(\mathbb{1}_{C_3}) \oplus \mathcal{IC}(\mathcal{F}_{C_3})$	

5.3. **Adams-Barbasch-Vogan packets.**

5.3.1. *Admissible representations versus equivariant perverse sheaves.*

$\mathsf{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}}$	$\Pi_{\text{pure},\lambda}(G/F)$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$[\pi(\phi_0),0]$
$\mathcal{IC}(\mathbb{1}_{C_2})$	$[\pi(\phi_2,+),0]$
$\mathcal{IC}(\mathbb{1}_{C_3})$	$[\pi(\phi_3,+),0]$
$\mathcal{IC}(\mathcal{L}_{C_3})$	$[\pi(\phi_3,-),0]$
$\mathcal{IC}(\mathcal{F}_{C_2})$	$[\pi(\phi_2,-),1]$

5.3.2. *ABV-packets.*

ABV-packets	packet representations	coronal representations
$\Pi^{\text{ABV}}_{\text{pure},\phi_0}(G/F):$	$[\pi(\phi_0,+),0]$	$[\pi(\phi_2,+),0]$
$\Pi^{\text{ABV}}_{\text{pure},\phi_2}(G/F):$	$[\pi(\phi_2,+),0],\ [\pi(\phi_2,-),1]$	$[\pi(\phi_3,-),0]$
$\Pi^{\text{ABV}}_{\text{pure},\phi_3}(G/F):$	$[\pi(\phi_3,+),0],\ [\pi(\phi_3,-),0]$	

5.3.3. *Stable distributions and endoscopic transfer.* We now calculate the virtual representations $\eta^{\mathsf{NEv}}_{\phi,s}$; see (29). In the list below, we use the notation $s=(s_1,s_2)$ for elements of $\widehat{T}[2]$.

ϕ_0:

$$\eta^{\mathsf{NEv}}_{\phi_0,s} = [\pi(\phi_0,+),0]+(-)(s_1s_2)[\pi(\phi_2,+),0]$$

so

$$\begin{aligned}
\eta^{\mathsf{NEv}}_{\phi_0,(1,1)} &= [\pi(\phi_0),0]+[\pi(\phi_2,+),0]\\
\eta^{\mathsf{NEv}}_{\phi_0,(1,-1)} &= [\pi(\phi_0),0]-[\pi(\phi_2,+),0]\\
\eta^{\mathsf{NEv}}_{\phi_0,(-1,1)} &= [\pi(\phi_0),0]-[\pi(\phi_2,+),0]\\
\eta^{\mathsf{NEv}}_{\phi_0,(-1,-1)} &= [\pi(\phi_0),0]+[\pi(\phi_2,+),0].
\end{aligned}$$

ϕ_2:

$$\eta^{\mathsf{NEv}}_{\phi_2,s} = [\pi(\phi_2,+),0]-(+-)(s)[\pi(\phi_2,-),1]-(--)(s)[\pi(\phi_3,-),0]$$

so

$$\begin{aligned}
\eta^{\mathsf{NEv}}_{\phi_2,(1,1)} &= [\pi(\phi_2,+),0]-[\pi(\phi_2,-),1]-[\pi(\phi_3,-),0]\\
\eta^{\mathsf{NEv}}_{\phi_2,(1,-1)} &= [\pi(\phi_2,+),0]+[\pi(\phi_2,-),1]+[\pi(\phi_3,-),0]\\
\eta^{\mathsf{NEv}}_{\phi_2,(-1,1)} &= [\pi(\phi_2,+),0]-[\pi(\phi_2,-),1]+[\pi(\phi_3,-),0]\\
\eta^{\mathsf{NEv}}_{\phi_2,(-1,-1)} &= [\pi(\phi_2,+),0]+[\pi(\phi_2,-),1]-[\pi(\phi_3,-),0]
\end{aligned}$$

ϕ_3:

$$\eta^{\mathsf{NEv}}_{\phi_3,s} = [\pi(\phi_3,+),0]+(-)(s_1s_2)[\pi(\phi_3,-),0]$$

so

$$\begin{aligned}
\eta^{\mathsf{NEv}}_{\phi_3,(1,1)} &= [\pi(\phi_3,+),0]+[\pi(\phi_3,-),0]\\
\eta^{\mathsf{NEv}}_{\phi_3,(1,-1)} &= [\pi(\phi_3,+),0]-[\pi(\phi_3,-),0]\\
\eta^{\mathsf{NEv}}_{\phi_3,(-1,1)} &= [\pi(\phi_3,+),0]-[\pi(\phi_3,-),0]\\
\eta^{\mathsf{NEv}}_{\phi_3,(-1,-1)} &= [\pi(\phi_3,+),0]+[\pi(\phi_3,-),0].
\end{aligned}$$

After comparing with Section 5.1.6, we see

$$\begin{aligned}
\eta_{\psi_0,s} &= \eta^{\mathsf{NEv}}_{\psi_0,s}\\
\eta_{\psi_3,s} &= \eta^{\mathsf{NEv}}_{\psi_3,s}.
\end{aligned}$$

This proves (28) for admissible represenations with infinitesimal parameter λ given at the beginning of Section 5.

5.3.4. *Kazhdan-Lusztig conjecture.* From Section 5.1.3 we find the multiplicity matrix m_{rep} and from Section 5.2.3 we find the normalised geometric multiplicity matrix m'_{geo}:

$$m_{\mathrm{rep}} = \left(\begin{array}{cccc|c} 1&1&1&1&0\\ 0&1&1&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ \hline 0&0&0&0&1 \end{array}\right), \qquad m'_{\mathrm{geo}} = \left(\begin{array}{cccc|c} 1&0&0&0&0\\ 1&1&0&0&0\\ 1&1&1&0&0\\ 1&0&0&1&0\\ \hline 0&0&0&0&1 \end{array}\right).$$

Since ${}^t m_{\mathrm{rep}} = m'_{\mathrm{geo}}$, this confirms the Kazhdan-Lusztig conjecture as it applies to representations with infinitesimal parameter $\lambda : W_F \to {}^L G$ given at the beginning of Section 5.

5.3.5. *Aubert duality and Fourier transform.* To verify (30), use Vogan's bijection from Section 5.3.1 to compare Aubert duality from Section 5.1.5 with the Fourier transform from Section 5.2.4

To verify (31), observe that the twisting characters χ_ψ of A_ψ from Section 5.1.5 are trivial except for the Arthur parameter ψ_2, as are the local systems $\mathcal{T}_\psi$ from Section 5.2.7 and in both cases they are given by the character $(--)$ of A_{ψ_2} determined by the isomorphism $A_{\psi_2} \cong \{\pm 1\} \times \{\pm 1\}$, fixed in Section 5.1.4.

5.4. **Endoscopy and equivariant restriction of perverse sheaves.** As in Section 5.1.6, we now consider the split endoscopic group $G' = \mathrm{SO}(3) \times \mathrm{SO}(3)$ for G determined by $s = \mathrm{diag}(1,-1,-1,1) \in \widehat{G}$. Then $\lambda : W_F \to {}^L G$ factors through $\epsilon : {}^L G' \to {}^L G$ to define $\lambda' : W_F \to {}^L G'$ by

$$\lambda'(w) = \left(\begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \right).$$

In this section we will calculate both sides of (34). This will illustrate how the Langlands-Shelstad lift of $\Theta_{\psi'}$ on $G'(F)$ to $\Theta_{\psi,s}$ on $G(F)$ is related to equivariant restriction of perverse sheaves from V to the Vogan variety V' for G'. Note that each component of λ' is the infinitesimal parameter $W_F \to {}^L\mathrm{SO}(3)$ that appeared in Section 2; here we will use that Section extensively.

5.4.1. *Parameters.* There are four Arthur parameters with infinitesimal parameter $\lambda' : W_F \to {}^L G'$, up to H'-conjugacy. Using notation from Section 2, they are

$$\begin{array}{rclrcl} \psi'_{00} & := & \psi_0 \times \psi_0, & \psi'_{11} & := & \psi_1 \times \psi_1, \\ \psi'_{10} & := & \psi_1 \times \psi_0, & \psi'_{01} & := & \psi_0 \times \psi_1, \end{array}$$

so

$$\begin{array}{rclrcl} \psi'_{00}(w,x,y) & = & (\nu_2(y), \nu_2(y)), & \psi'_{11}(w,x,y) & = & (\nu_2(x), \nu_2(x)), \\ \psi'_{10}(w,x,y) & = & (\nu_2(x), \nu_2(y)), & \psi'_{01}(w,x,y) & = & (\nu_2(y), \nu_2(x)). \end{array}$$

Although $\psi_2 = \epsilon \circ \psi'_{10}$ is H-conjugate to $\epsilon \circ \psi'_{01}$, the Arthur parameters ψ'_{10} and ψ'_{01} for G' are not H'-conjugate.

5.4.2. *Endoscopic Vogan variety.* The Vogan variety V' for λ' is simply two copies of the Vogan variety appearing in Section 2. As a subvariety of the conormal bundle to V, the conormal to the Vogan variety V' for $\lambda' : W_F \to {}^L G'$ is

$$T^*_{H'}(V') = \left\{ \left(\begin{array}{cc|cc} & & 0 & x \\ & & y & 0 \\ \hline 0 & y' & & \\ x' & 0 & & \end{array} \right) \;\middle|\; \begin{array}{l} xy' = 0 \\ yx' = 0 \end{array} \right\}$$

(C'_0). Set $C'_0 = C_0 \times C_0$. Then the regular conormal above the closed H'-orbit $C'_0 \subset V'$ is

$$T^*_{C'_0}(V')_{\mathrm{reg}} = \left\{ \left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & y' & & \\ x' & 0 & & \end{array} \right) \;\middle|\; \begin{array}{l} x' \neq 0 \\ y' \neq 0 \end{array} \right\}$$

Base point:

$$(x'_0, \xi'_0) = \left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 1 & & \\ 1 & 0 & & \end{array} \right) \in T^*_{C'_0}(V')_{\mathrm{reg}}$$

Fundamental groups:

$$\begin{array}{ccccc} & & \widehat{T}[2] & & \\ & & \big\downarrow{\scriptstyle s \mapsto (s_1, s_2)} & & \\ 1 = A_{x'_0} & \longleftarrow & A_{(x'_0, \xi'_0)} & \xrightarrow{\mathrm{id}} & A_{\xi'_0} = \{\pm 1\} \times \{\pm 1\} \end{array}$$

Local systems on strongly regular conormal:

$\mathsf{Loc}_H(T^*_{C'_0}(V)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}'_0}$	$\mathcal{L}_{\mathcal{O}'_0}$	$\mathcal{F}_{\mathcal{O}'_0}$	$\mathcal{E}_{\mathcal{O}'_0}$
$\mathsf{Rep}(A_{(x'_0, \xi'_0)})$:	$++$	$--$	$-+$	$+-$

Pullback along the bundle map:

$$\begin{array}{ccc} \mathsf{Loc}_{H'}(C'_0) & \to & \mathsf{Loc}_{H'}(T^*_{C'_0}(V')_{\mathrm{sreg}}) \\ \mathbb{1}_{C'_0} & \mapsto & \mathbb{1}_{\mathcal{O}'_0} \\ & & \mathcal{L}_{\mathcal{O}'_0} \\ & & \mathcal{F}_{\mathcal{O}'_0} \\ & & \mathcal{E}_{\mathcal{O}'_0} \end{array}$$

(C'_x). Set $C'_x = C_x \times C_0 \subset V'$. Then the regular conormal above C'_x is

$$T^*_{C'_x}(V)_{\mathrm{reg}} = \left\{ \left(\begin{array}{cc|cc} & & 0 & x \\ & & 0 & 0 \\ \hline 0 & y' & & \\ 0 & 0 & & \end{array} \right) \;\middle|\; \begin{array}{c} x \neq 0 \\ y' \neq 0 \end{array} \right\}$$

Base point:

$$(x'_{10}, \xi'_{10}) = \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 0 & 0 \\ \hline 0 & 1 & & \\ 0 & 0 & & \end{array} \right) \in T^*_{C'_x}(V)_{\mathrm{reg}}$$

Fundamental groups:

$$\begin{array}{ccccc} & & \widehat{T}[2] & & \\ & & \big\downarrow{\scriptstyle s \mapsto (s_1,s_2)} & & \\ \{\pm 1\} = A_{x'_{10}} & \xleftarrow{s_1 \leftarrow\!\shortmid (s_1,s_2)} & A_{(x'_{10},\xi'_{10})} & \xrightarrow{(s_1,s_2)\mapsto s_2} & A_{\xi'_{10}} = \{\pm 1\} \end{array}$$

Local systems on strongly regular conormal:

$\mathsf{Loc}_H(T^*_{C'_x}(V)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}'_x}$	$\mathcal{L}_{\mathcal{O}'_x}$	$\mathcal{F}_{\mathcal{O}'_x}$	$\mathcal{E}_{\mathcal{O}'_x}$
$\mathsf{Rep}(A_{(x'_{01},\xi'_{01})})$:	$++$	$--$	$-+$	$+-$

Pullback along the bundle map:

$$\begin{array}{ccc} \mathsf{Loc}_{H'}(C'_x) & \to & \mathsf{Loc}_{H'}(T^*_{C'_x}(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C'_x} & \mapsto & \mathbb{1}_{\mathcal{O}'_x} \\ & & \mathcal{L}_{\mathcal{O}'_x} \\ \mathcal{L}_{C'_x} & \mapsto & \mathcal{F}_{\mathcal{O}'_x} \\ & & \mathcal{E}_{\mathcal{O}'_x} \end{array}$$

(C'_y). Set $C'_y = C_0 \times C_x \subset V'$. Then the regular conormal above C'_y is

$$T^*_{C'_y}(V)_{\mathrm{reg}} = \left\{ \left(\begin{array}{cc|cc} & & 0 & 0 \\ & & y & 0 \\ \hline 0 & 0 & & \\ x' & 0 & & \end{array} \right) \;\middle|\; \begin{array}{c} x' \neq 0 \\ y \neq 0 \end{array} \right\}$$

Base point:

$$(x'_{01}, \xi'_{01}) = \left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 1 & 0 \\ \hline 0 & 0 & & \\ 1 & 0 & & \end{array} \right) \in T^*_{C'_y}(V)_{\mathrm{reg}}$$

Fundamental groups:

$$\begin{array}{ccccc} & & \widehat{T}[2] & & \\ & & \big\downarrow{\scriptstyle s \mapsto (s_1,s_2)} & & \\ \{\pm 1\} = A_{x'_{01}} & \xleftarrow{s_2 \leftarrow\!\shortmid (s_1,s_2)} & A_{(x'_{01},\xi'_{01})} & \xrightarrow{(s_1,s_2)\mapsto s_1} & A_{\xi'_{01}} = \{\pm 1\} \end{array}$$

Local systems on strongly regular conormal:

$\mathsf{Loc}_H(T^*_{C'_y}(V)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}'_y}$	$\mathcal{L}_{\mathcal{O}'_y}$	$\mathcal{F}_{\mathcal{O}'_y}$	$\mathcal{E}_{\mathcal{O}'_y}$
$\mathsf{Rep}(A_{(x'_{10},\xi'_{10})})$:	$++$	$--$	$-+$	$+-$

Pullback along the bundle map:

$$\begin{array}{ccc} \mathsf{Loc}_H(C'_y) & \to & \mathsf{Loc}_H(T^*_{C'_y}(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C'_y} & \mapsto & \mathbb{1}_{\mathcal{O}'_y} \\ & & \mathcal{L}_{\mathcal{O}'_y} \\ & & \mathcal{F}_{\mathcal{O}'_y} \\ \mathcal{L}_{C'_y} & \mapsto & \mathcal{E}_{\mathcal{O}'_y} \end{array}$$

(C'_{xy}). Set $C'_{xy} = C_x \times C_y \subset V'$. Then

$$T^*_{C_{xy}}(V)_{\mathrm{reg}} = \left\{ \left(\begin{array}{cc|cc} & & 0 & x \\ & & y & 0 \\ \hline 0 & 0 & & \\ 0 & 0 & & \end{array} \right) \;\middle|\; xy \neq 0 \right\}$$

Base point:

$$(x'_{11}, \xi'_{11}) = \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 1 & 0 \\ \hline 0 & 0 & & \\ 0 & 0 & & \end{array} \right) \in T^*_{C'_{xy}}(V)_{\mathrm{reg}}$$

Fundamental groups:

$$\begin{array}{ccccc} & & \widehat{T}[2] & & \\ & & \downarrow{\scriptstyle s \mapsto (s_1, s_2)} & & \\ \{\pm 1\} \times \{\pm 1\} = A_{x'_{11}} & \xleftarrow{\mathrm{id}} & A_{(x'_{11},\xi'_{11})} & \longrightarrow & A_{\xi'_{11}} = 1 \end{array}$$

Local systems on strongly regular conormal:

$\mathsf{Loc}_H(T^*_{C'_{xy}}(V)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}'_{xy}}$	$\mathcal{L}_{\mathcal{O}'_{xy}}$	$\mathcal{F}_{\mathcal{O}'_{xy}}$	$\mathcal{E}_{\mathcal{O}'_{xy}}$
$\mathsf{Rep}(A_{(x'_{11},\xi'_{11})})$:	$++$	$--$	$-+$	$+-$

Pullback along the bundle map:

$$\begin{array}{ccc} \mathsf{Loc}_{H'}(C'_{xy}) & \to & \mathsf{Loc}_H(T^*_{C'_{xy}}(V')_{\mathrm{sreg}}) \\ \mathbb{1}_{C'_{xy}} & \mapsto & \mathbb{1}_{\mathcal{O}'_{xy}} \\ \mathcal{L}_{C'_{xy}} & \mapsto & \mathcal{L}_{\mathcal{O}'_{xy}} \\ \mathcal{F}_{C'_{xy}} & \mapsto & \mathcal{F}_{\mathcal{O}'_{xy}} \\ \mathcal{E}_{C'_{xy}} & \mapsto & \mathcal{E}_{\mathcal{O}'_{xy}} \end{array}$$

5.4.3. *Vanishing cycles.* The functor

$${}^p\mathsf{NEv}' : \mathsf{Per}_{H'}(V') \to \mathsf{Per}_{H'}(T^*_{H'}(V')_{\mathrm{reg}})$$

may be deduced from Section 2.2.6 using the Sebastiani-Thom isomorphism [17]; see Table 5.4.3.1. Here we show the calculation of the last three rows, to illustrate the

method.

$$\begin{aligned}
{}^p\mathsf{NEv}'\,\mathcal{IC}(\mathcal{L}_{C'_x}) &= {}^p\mathsf{NEv}'\,(\mathcal{IC}(\mathcal{E}_{C_x})\boxtimes\mathcal{IC}(\mathbb{1}_{C_0}))\\
&= ({}^p\mathsf{NEv}\,\mathcal{IC}(\mathcal{E}_{C_x}))\boxtimes({}^p\mathsf{NEv}\,\mathcal{IC}(\mathbb{1}_{C_0}))\\
&= (\mathcal{IC}(\mathcal{E}_{\mathcal{O}_x})\oplus\mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}))\boxtimes\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})\\
&= \mathcal{IC}(\mathcal{E}_{\mathcal{O}_x})\boxtimes\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})\oplus\mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})\boxtimes\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})\\
&= \mathcal{IC}(\mathcal{E}_{\mathcal{O}_x}\boxtimes\mathbb{1}_{\mathcal{O}_0})\oplus\mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}\boxtimes\mathbb{1}_{\mathcal{O}_0})\\
&= \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_x})\oplus\mathcal{IC}(\mathcal{F}_{\mathcal{O}'_0})
\end{aligned}$$

Similarly,

$$\begin{aligned}
{}^p\mathsf{NEv}'\,\mathcal{IC}(\mathcal{L}_{C'_y}) &= {}^p\mathsf{NEv}'\,\big(\mathcal{IC}(\mathbb{1}_{C_0})\boxtimes\mathcal{IC}(\mathcal{E}_{C_y})\big)\\
&= ({}^p\mathsf{NEv}\,\mathcal{IC}(\mathbb{1}_{C_0}))\boxtimes\big({}^p\mathsf{NEv}\,\mathcal{IC}(\mathcal{E}_{C_y})\big)\\
&= \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})\boxtimes\big(\mathcal{IC}(\mathcal{E}_{\mathcal{O}_y})\oplus\mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})\big)\\
&= \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})\boxtimes\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})\oplus\mathcal{IC}(\mathcal{E}_{\mathcal{O}_y})\boxtimes\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})\\
&= \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}\boxtimes\mathbb{1}_{\mathcal{O}_0})\oplus\mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}\boxtimes\mathbb{1}_{\mathcal{O}_0})\\
&= \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_0})\oplus\mathcal{IC}(\mathcal{E}_{\mathcal{O}'_y})
\end{aligned}$$

and

$$\begin{aligned}
{}^p\mathsf{NEv}'\,\mathcal{IC}(\mathcal{L}_{C'_{xy}}) &= {}^p\mathsf{NEv}'\,\big(\mathcal{IC}(\mathcal{E}_{C_x})\boxtimes\mathcal{IC}(\mathcal{E}_{C_y})\big)\\
&= ({}^p\mathsf{NEv}\,\mathcal{IC}(\mathcal{E}_{C_x}))\boxtimes\big({}^p\mathsf{NEv}\,\mathcal{IC}(\mathcal{E}_{C_y})\big)\\
&= (\mathcal{IC}(\mathcal{E}_{\mathcal{O}_x})\oplus\mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}))\boxtimes\big(\mathcal{IC}(\mathcal{E}_{\mathcal{O}_y})\oplus\mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})\big)\\
&= \mathcal{IC}(\mathcal{E}_{\mathcal{O}_x}\boxtimes\mathcal{E}_{\mathcal{O}_y})\oplus\mathcal{IC}(\mathcal{E}_{\mathcal{O}_x}\boxtimes\mathcal{E}_{\mathcal{O}_0})\oplus\mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}\boxtimes\mathcal{E}_{\mathcal{O}_y})\oplus\mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}\boxtimes\mathcal{E}_{\mathcal{O}_0})\\
&= \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_{xy}})\oplus\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x})\oplus\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y})\oplus\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0}).
\end{aligned}$$

5.4.4. *Restriction.*

$$\begin{aligned}
\mathrm{res}:\mathsf{Per}_H(V) &\longrightarrow \mathsf{KPer}_{H'}(V')\\
\mathcal{IC}(\mathbb{1}_{C_0}) &\mapsto \mathcal{IC}(\mathbb{1}_{C'_0})\\
\mathcal{IC}(\mathbb{1}_{C_2}) &\mapsto \mathcal{IC}(\mathbb{1}_{C'_x})[1]\oplus\mathcal{IC}(\mathbb{1}_{C'_y})[1]\oplus\mathcal{IC}(\mathbb{1}_{C'_0})[1]\\
\mathcal{IC}(\mathbb{1}_{C_3}) &\mapsto \mathcal{IC}(\mathbb{1}_{C'_{xy}})[1]\\
\mathcal{IC}(\mathcal{L}_{C_3}) &\mapsto \mathcal{IC}(\mathbb{1}_{C'_0})[1]\oplus\mathcal{IC}(\mathcal{L}_{C'_{xy}})[1]\\
\mathcal{IC}(\mathcal{F}_{C_2}) &\mapsto \mathcal{IC}(\mathcal{L}_{C'_x})[1]\oplus\mathcal{IC}(\mathcal{L}_{C'_y})[1]
\end{aligned}$$

5.4.5. *Restriction and vanishing cycles.* In this example the inclusion $V'\hookrightarrow V$ induces a map of conormal bundles $\epsilon: T^*_{H'}(V')\hookrightarrow T^*_H(V)$; this is not true in general, as Section 6.4.3 shows. Here we have

$$\begin{aligned}
T^*_{C_0}(V)_{\mathrm{reg}}\cap T^*_{H'}(V')_{\mathrm{reg}} &= T^*_{C'_0}(V')_{\mathrm{reg}}\\
T^*_{C_2}(V)_{\mathrm{reg}}\cap T^*_{H'}(V')_{\mathrm{reg}} &= T^*_{C'_x}(V')_{\mathrm{reg}}\cup T^*_{C'_y}(V')_{\mathrm{reg}}\\
T^*_{C_3}(V)_{\mathrm{reg}}\cap T^*_{H'}(V')_{\mathrm{reg}} &= T^*_{C'_{xy}}(V')_{\mathrm{reg}}
\end{aligned}$$

We now calculate both sides of (33) in three cases: when $\mathcal{P}=\mathcal{IC}(\mathbb{1}_{C_2})$, when $\mathcal{P}=\mathcal{IC}(\mathcal{L}_{C_3})$ and when $\mathcal{P}=\mathcal{IC}(\mathcal{F}_{C_2})$.

($\mathcal{IC}(\mathbb{1}_{C_2})$). Take $\mathcal{P}=\mathcal{IC}(\mathbb{1}_{C_2})$. By Section 5.4.4,

$$\mathcal{IC}(\mathbb{1}_{C_2})|_{V'}\equiv\mathcal{IC}(\mathbb{1}_{C'_x})[1]\oplus\mathcal{IC}(\mathbb{1}_{C'_y})[1]\oplus\mathcal{IC}(\mathbb{1}_{C'_0})[1],$$

TABLE 5.4.3.1. ${}^p\mathsf{NEv}' : \mathsf{Per}_{H_{\lambda'}}(V_{\lambda'}) \to \mathsf{Per}_{H_{\lambda'}}(T^*_{H_{\lambda'}}(V_{\lambda'})_{\mathrm{reg}})$ on simple objects, for $\lambda' : W_F \to {}^LG'$ given at the beginning of Section 5.

$$
\begin{array}{ccl}
\mathsf{Per}_{H'}(V') & \xrightarrow{{}^p\mathsf{NEv}'} & \mathsf{Per}_{H'}(T^*_{H'}(V')_{\mathrm{reg}}) \\
\mathcal{IC}(\mathbb{1}_{C'_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}'_0}) \\
\mathcal{IC}(\mathbb{1}_{C'_x}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}'_x}) \\
\mathcal{IC}(\mathbb{1}_{C'_y}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}'_y}) \\
\mathcal{IC}(\mathbb{1}_{C'_{xy}}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}'_{xy}}) \\
\mathcal{IC}(\mathcal{L}_{C'_x}) & \mapsto & \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_x}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_0}) \\
\mathcal{IC}(\mathcal{L}_{C'_y}) & \mapsto & \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_y}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_0}) \\
\mathcal{IC}(\mathcal{L}_{C'_{xy}}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_{xy}}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0})
\end{array}
$$

$\mathcal{P}'$	$\mathsf{NEvs}_{\psi'_{00}}\mathcal{P}'$	$\mathsf{NEvs}_{\psi'_{10}}\mathcal{P}'$	$\mathsf{NEvs}_{\psi'_{01}}\mathcal{P}'$	$\mathsf{NEvs}_{\psi'_{11}}\mathcal{P}'$
$\mathcal{IC}(\mathbb{1}_{C'_0})$	$++$	0	0	0
$\mathcal{IC}(\mathbb{1}_{C'_x})$	0	$++$	0	0
$\mathcal{IC}(\mathbb{1}_{C'_y})$	0	0	$++$	0
$\mathcal{IC}(\mathbb{1}_{C'_{xy}})$	0	0	0	$++$
$\mathcal{IC}(\mathcal{L}_{C'_x})$	$-+$	$-+$	0	0
$\mathcal{IC}(\mathcal{L}_{C'_y})$	$+-$	0	$+-$	0
$\mathcal{IC}(\mathcal{L}_{C'_{xy}})$	$--$	$--$	$--$	$--$

after passing to the Grothendieck group of $\mathsf{Per}_{H'}(V')$. So, by Section 5.4.3,

$$
\begin{aligned}
& \mathsf{NEv}'\left(\mathcal{IC}(\mathbb{1}_{C_2})|_{V'}\right) \\
\equiv\ & \mathsf{NEv}'\left(\mathcal{IC}(\mathbb{1}_{C'_x})[1] \oplus \mathcal{IC}(\mathbb{1}_{C'_y})[1] \oplus \mathcal{IC}(\mathbb{1}_{C'_0})[1]\right) \\
=\ & \mathcal{IC}(\mathbb{1}_{\mathcal{O}'_x})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0})[1]
\end{aligned}
$$

in the Grothendieck group of $\mathsf{Per}_{H'}(T^*_{H'}(V')_{\mathrm{reg}})$. Thus, for each $(x', \xi') \in T^*_{C'}(V')_{\mathrm{reg}}$ with image $(x, \xi) \in T^*_C(V)_{\mathrm{reg}}$, the left-hand side of (33) is

$$
\begin{aligned}
& (-1)^{\dim C'} \operatorname{trace}_{a'_s}\left(\mathsf{NEv}'\,\mathcal{IC}(\mathbb{1}_{C_2})|_{V'}\right)_{(x',\xi')} \\
=\ & (-1)^{\dim C'} \operatorname{trace}_{a'_s}\left(\mathcal{IC}(\mathbb{1}_{\mathcal{O}'_x})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0})[1]\right)_{(x',\xi')}
\end{aligned}
$$

while the right-hand side of (33) is

$$
\begin{aligned}
& (-1)^{\dim C} \operatorname{trace}_{a_s}(\mathsf{Ev}\,\mathcal{IC}(\mathbb{1}_{C_2}))_{(x,\xi)} \\
=\ & (-1)^{\dim C} \operatorname{trace}_{a_s}\left(\mathcal{IC}(\mathbb{1}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})\right)_{(x,\xi)}.
\end{aligned}
$$

We now calculate both sides of (33) when $\mathcal{P} = \mathcal{IC}(\mathbb{1}_{C_2})$.

(C'_0). If $(x', \xi') \in T^*_{C'_0}(V')_{\mathrm{reg}}$ then $C' = C'_0$ and $C = C_0$ and the left-hand side of (33) is

$$
\begin{aligned}
& (-1)^{\dim C'_0} \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathbb{1}_{\mathcal{O}'_0})[1] \\
=\ & -(-1)^0 \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathbb{1}_{\mathcal{O}'_0}) \\
=\ & -(++)(+1,-1) \\
=\ & -1,
\end{aligned}
$$

while the right-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C_0}\operatorname{trace}_{(+1,-1)}\left(\mathcal{IC}(\mathbb{1}_{\mathcal{O}_2})\oplus\mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})\right)|_{T^*_{C_0}(V)_{\mathrm{reg}}}\\
&= \operatorname{trace}_{(+1,-1)}\mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})\\
&= (--)(+1,-1)\\
&= -1.
\end{aligned}$$

This confirms (33) on $T^*_{C'_0}(V')_{\mathrm{reg}}$.

(C'_x). If $(x',\xi')\in T^*_{C'_x}(V')_{\mathrm{reg}}$ then $C'=C'_x$ and $C=C_2$ and the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C'_x}\operatorname{trace}_{(+1,-1)}\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x})[1]\\
&= -(-1)^1\operatorname{trace}_{(+1,-1)}\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x})\\
&= (--)(+1,-1)\\
&= -1
\end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C_2}\operatorname{trace}_{(-1)}\left(\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3})\oplus\mathcal{IC}(\mathcal{L}_{\mathcal{O}_2})\right)|_{T^*_{C_2}(V)_{\mathrm{reg}}}\\
&= \operatorname{trace}_{(-1)}\mathcal{IC}(\mathcal{L}_{\mathcal{O}_2})\\
&= (-)(-1)\\
&= -1.
\end{aligned}$$

This confirms (33) on $T^*_{C'_x}(V')_{\mathrm{reg}}$.

(C'_y). If $(x',\xi')\in T^*_{C'_y}(V')_{\mathrm{reg}}$ then $C'=C'_y$ and $C=C_2$ and the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C'_y}\operatorname{trace}_{(-1,+1)}\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y})[1]\\
&= -(-1)^1\operatorname{trace}_{(-1,+1)}\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y})\\
&= (--)(-1,+1)\\
&= -1,
\end{aligned}$$

while the right-hand side of (33) is -1, as in the case above. This confirms (33) on $T^*_{C'_y}(V')_{\mathrm{reg}}$.

(C'_{xy}). If $(x',\xi')\in T^*_{C'_{xy}}(V')_{\mathrm{reg}}$ then $C'=C'_{xy}$ and $C=C_3$ and the left-hand side of (33) is

$$(-1)^{\dim C'_{xy}}\operatorname{trace}_{(+1,-1)}\left(\mathcal{IC}(\mathbb{1}_{\mathcal{O}'_x})[1]\oplus\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y})[1]\oplus\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0})[1]\right)_{(x',\xi')}=0$$

while the right-hand side of (33) is

$$(-1)^{\dim C_3}\operatorname{trace}_{(-1)}\left(\mathcal{IC}(\mathbb{1}_{\mathcal{O}_2})\oplus\mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})\right)|_{T^*_{C_3}(V)_{\mathrm{reg}}}=0.$$

This confirms (33) on $T^*_{C'_{xy}}(V')_{\mathrm{reg}}$.

This proves (34) when $\mathcal{P}=\mathcal{IC}(\mathbb{1}_{C_2})$.

($\mathcal{IC}(\mathcal{L}_{C_3})$). We now calculate both sides of (33) when $\mathcal{P}=\mathcal{IC}(\mathcal{L}_{C_3})$. By Section 5.4.4,

$$\mathcal{IC}(\mathcal{L}_{C_3})|_{V'}\equiv\mathcal{IC}(\mathbb{1}_{C'_0})[1]\oplus\mathcal{IC}(\mathcal{L}_{C'_{xy}})[1],$$

after passing to the Grothendieck group of $\mathsf{Per}_{H'}(V')$. So, by Section 5.4.3,

$$\begin{aligned}
&\mathsf{NEv}'\left(\mathcal{IC}(\mathcal{L}_{C_3})|_{V'}\right)\\
&\equiv \mathsf{NEv}'\left(\mathcal{IC}(\mathbb{1}_{C'_0})[1]\oplus\mathcal{IC}(\mathcal{L}_{C'_{xy}})[1]\right)\\
&= \mathcal{IC}(\mathbb{1}_{\mathcal{O}'_0})[1]\oplus\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_{xy}})[1]\oplus\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y})[1]\oplus\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x})[1]\oplus\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0})[1]
\end{aligned}$$

in the Grothendieck group of $\mathsf{Per}_{H'}(T^*_{H'}(V')_{\mathrm{reg}})$. Thus, for each $(x',\xi') \in T^*_{C'}(V')_{\mathrm{reg}}$ with image $(x,\xi) \in T^*_C(V)_{\mathrm{reg}}$, the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C'} \operatorname{trace}_{a'_s} \left(\mathsf{NEv}'\, \mathcal{IC}(\mathcal{L}_{C_3})|_{V'}\right)_{(x',\xi')} \\
=\ & (-1)^{\dim C'} \operatorname{trace}_{a'_s} \left(\mathcal{IC}(\mathbb{1}_{\mathcal{O}'_0})[1]\right)_{(x',\xi')} \\
&+(-1)^{\dim C'} \operatorname{trace}_{a'_s} \left(\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_{xy}})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0})[1]\right)_{(x',\xi')}
\end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C} \operatorname{trace}_{a_s} (\mathsf{Ev}\, \mathcal{IC}(\mathcal{L}_{C_3}))_{(x,\xi)} \\
=\ & (-1)^{\dim C} \operatorname{trace}_{a_s} \left(\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2})\right)_{(x,\xi)}.
\end{aligned}$$

We now calculate both sides of (33) in every case.

(C'_0). If $(x',\xi') \in T^*_{C'_0}(V')_{\mathrm{reg}}$ then $C' = C'_0$ and $C = C_0$ and the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C'_0} \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathbb{1}_{\mathcal{O}'_0})[1] + (-1)^{\dim C'_0} \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0})[1] \\
=\ & -(-1)^0 \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathbb{1}_{\mathcal{O}'_0}) - (-1)^0 \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0}) \\
=\ & -(++)(+1,-1) - (--)(+1,-1) \\
=\ & +1-1 \\
=\ & 0,
\end{aligned}$$

while the right-hand side of (33) is

$$(-1)^{\dim C_0} \operatorname{trace}_{(-1)} \left(\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2})\right)|_{T^*_{C_0}(V)_{\mathrm{reg}}} = 0.$$

This confirms (33) on $T^*_{C'_0}(V')_{\mathrm{reg}}$.

(C'_x). If $(x',\xi') \in T^*_{C'_x}(V')_{\mathrm{reg}}$ then $C' = C'_x$ and $C = C_2$ and the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C'_x} \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x})[1] \\
=\ & -(-1)^1 \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x}) \\
=\ & (--)(+1,-1) \\
=\ & -1
\end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C_2} \operatorname{trace}_{(-1)} \left(\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2})\right)|_{T^*_{C_2}(V)_{\mathrm{reg}}} \\
=\ & \operatorname{trace}_{(-1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2}) \\
=\ & (-)(-1) \\
=\ & -1.
\end{aligned}$$

This confirms (33) on $T^*_{C'_x}(V')_{\mathrm{reg}}$.

(C'_y). If $(x',\xi') \in T^*_{C'_y}(V')_{\mathrm{reg}}$ then $C' = C'_y$ and $C = C_2$ and the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C'_y} \operatorname{trace}_{(-1,+1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y})[1] \\
=\ & -(-1)^1 \operatorname{trace}_{(-1,+1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y}) \\
=\ & (--)(-1,+1) \\
=\ & -1,
\end{aligned}$$

while the right-hand side of (33) is -1, as in the case above. This confirms (33) on $T^*_{C'_y}(V')_{\mathrm{reg}}$.

(C'_{xy}). If $(x',\xi') \in T^*_{C'_{xy}}(V')_{\mathrm{reg}}$ then $C' = C'_{xy}$ and $C = C_3$ and the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C'_{xy}} \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_{xy}})[1] \\
&= -(-1)^2 \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_{xy}}) \\
&= -(--)(+1,-1) \\
&= -(-1) \\
&= +1
\end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C_3} \operatorname{trace}_{(-1)} \left(\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2})\right)|_{T^*_{C_3}(V)_{\mathrm{reg}}} \\
&= -\operatorname{trace}_{(-1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \\
&= -(-)(-1) \\
&= +1.
\end{aligned}$$

This confirms (33) on $T^*_{C'_{xy}}(V')_{\mathrm{reg}}$.

This proves (34) when $\mathcal{P} = \mathcal{IC}(\mathcal{L}_{C_3})$.

($\mathcal{IC}(\mathcal{F}_{C_2})$). The case $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_2})$ gives another interesting example of (33). By Section 5.4.4,

$$\mathcal{IC}(\mathcal{F}_{C_2})|_{V'} \equiv \mathcal{IC}(\mathbb{1}_{C'_x})[1] \oplus \mathcal{IC}(\mathcal{L}_{C'_y})[1],$$

after passing to the Grothendieck group of $\mathsf{Per}_{H'}(V')$. So, by Section 5.4.3,

$$\begin{aligned}
&\mathsf{NEv}'\left(\mathcal{IC}(\mathcal{F}_{C_2})|_{V'}\right) \\
&\equiv \mathsf{NEv}'\left(\mathcal{IC}(\mathcal{L}_{C'_x})[1] \oplus \mathcal{IC}(\mathcal{L}_{C'_y})[1]\right) \\
&= \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_x})[1] \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_0})[1] \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_y})[1] \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_0})[1]
\end{aligned}$$

in the Grothendieck group of $\mathsf{Per}_{H'}(T^*_{H'}(V')_{\mathrm{reg}})$. Thus, for each $(x',\xi') \in T^*_{C'}(V')_{\mathrm{reg}}$ with image $(x,\xi) \in T^*_C(V)_{\mathrm{reg}}$, the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C'} \operatorname{trace}_{a'_s} \left(\mathsf{NEv}'\,\mathcal{IC}(\mathcal{F}_{C_2})|_{V'}\right)_{(x',\xi')} \\
&= (-1)^{\dim C'} \operatorname{trace}_{a'_s} \left(\mathcal{IC}(\mathcal{F}_{\mathcal{O}'_x})[1] \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_0})[1]\right)_{(x',\xi')} \\
&\quad + (-1)^{\dim C'} \operatorname{trace}_{a'_s} \left(\mathcal{IC}(\mathcal{E}_{\mathcal{O}'_y})[1] \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_0})[1]\right)_{(x',\xi')}
\end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C} \operatorname{trace}_{a_s} (\mathsf{Ev}_{(x,\xi)}\,\mathcal{IC}(\mathcal{F}_{C_2})) \\
&= (-1)^{\dim C} \operatorname{trace}_{a_s} \left(\mathsf{Ev}\,\mathcal{IC}(\mathcal{F}_{C_2})\right)|_{T^*_C(V)_{\mathrm{reg}}} \\
&= (-1)^{\dim C} \operatorname{trace}_{a_s} \left(\mathcal{IC}(\mathcal{F}_{\mathcal{O}_2})\right)|_{T^*_C(V)_{\mathrm{reg}}}.
\end{aligned}$$

We now calculate both sides of (33) in every case.

(C'_0). If $(x',\xi') \in T^*_{C'_0}(V')_{\mathrm{reg}}$ then $C' = C'_0$ and $C = C_0$ and the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C'_0} \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_0})[1] + (-1)^{\dim C'_0} \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_0})[1] \\
&= -(-1)^0 \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_0}) - (-1)^0 \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_0}) \\
&= -(-+)(+1,-1) - (+-)(+1,-1) \\
&= +1 - 1 \\
&= 0,
\end{aligned}$$

while the right-hand side of (33) is

$$(-1)^{\dim C_0} \operatorname{trace}_{(-1)} \left(\mathcal{IC}(\mathcal{F}_{\mathcal{O}_2})\right)|_{T^*_{C_0}(V)_{\mathrm{reg}}} = 0.$$

This confirms (33) on $T^*_{C'_0}(V')_{\mathrm{reg}}$.

(C'_x). If $(x',\xi') \in T^*_{C'_x}(V')_{\mathrm{reg}}$ then $C' = C'_x$ and $C = C_2$ and the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C'_x} \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_x})[1] \\
&= -(-1)^1 \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_x}) \\
&= (-+)(+1,-1) \\
&= +1,
\end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C_2} \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}_2})|_{T^*_{C_2}(V)_{\mathrm{reg}}} \\
&= \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}_2}) \\
&= (-+)(+1,-1) \\
&= +1.
\end{aligned}$$

This confirms (33) on $T^*_{C'_x}(V')_{\mathrm{reg}}$.

(C'_y). If $(x',\xi') \in T^*_{C'_y}(V')_{\mathrm{reg}}$ then $C' = C'_y$ and $C = C_2$ and the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C'_y} \operatorname{trace}_{(-1,+1)} \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_y})[1] \\
&= -(-1)^1 \operatorname{trace}_{(-1,+1)} \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_x}) \\
&= (+-)(-1,+1) \\
&= +1,
\end{aligned}$$

while the right-hand side of (33) is $+1$, as in the case above. This confirms (33) on $T^*_{C'_y}(V')_{\mathrm{reg}}$.

(C'_{xy}). If $(x',\xi') \in T^*_{C'_{xy}}(V')_{\mathrm{reg}}$ then $C' = C'_{xy}$ and $C = C_3$ and the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C'_{xy}} \operatorname{trace}_{(+1,-1)} \left(\mathcal{IC}(\mathcal{F}_{\mathcal{O}'_x})[1] \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_0})[1]\right)_{(x',\xi')} \\
&\quad 0
\end{aligned}$$

while the right-hand side of (33) is

$$(-1)^{\dim C} \operatorname{trace}_{a_s} \left(\mathcal{IC}(\mathcal{F}_{\mathcal{O}_2})\right)|_{T^*_{C_3}(V)_{\mathrm{reg}}} = 0.$$

This confirms (33) on $T^*_{C'_{xy}}(V')_{\mathrm{reg}}$.

This proves (34) when $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_2})$.

6. SO(7) unipotent representations, singular infinitesimal parameter

Let $G = \mathrm{SO}(7)$. The calculation of pure inner twists and inner twists and their forms for G is the same as in Section 4. Let $G_0 = G$ be the split form of G and let G_1 be the non-quasisplit form of G, given by the quadratic form

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -\varepsilon\varpi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \varepsilon & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \varpi & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

One readily verifies that the Hasse invariant of this form is $(\varpi, \varepsilon) = -1$ so that the form is not split. Note that the choice $\varepsilon = 1$ would give a split form.

Consider the infinitesimal parameter $\lambda : W_F \to \widehat{G}$ given by

$$\lambda(w) := \begin{pmatrix} |w|^{3/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & |w|^{1/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & |w|^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & |w|^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & |w|^{-1/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & |w|^{-3/2} \end{pmatrix}.$$

Here, and below, we use the symplectic form $\langle x, y\rangle = {}^t x J y$ with matrix J given by $J_{ij} = (-1)^j \delta_{7-i,j}$ to determine a representation of $\mathrm{Sp}(6)$. Note that, in contrast to the unramified infinitesimal parameters in Sections 2 and 4, in this case the image of Frobenius is singular semisimple.

6.1. **Arthur packets.**

6.1.1. *Parameters.* Up to H_λ-conjugation, there are eight Langlands parameters with infinitesimal parameter λ, of which six are of Arthur type. The six Langlands parameters of Arthur type are most easily described through their Arthur parameters:

$$\begin{aligned} \psi_0(w,x,y) &= \nu_4(y) \oplus \nu_2(y), & \psi_7(w,x,y) &= \nu_4(x) \oplus \nu_2(x), \\ \psi_2(w,x,y) &= \nu_4(y) \oplus \nu_2(x), & \psi_6(w,x,y) &= \nu_4(x) \oplus \nu_2(y), \\ \psi_4(w,x,y) &= \nu_2(x) \otimes \nu_3(y), & \psi_5(w,x,y) &= \nu_3(x) \otimes \nu_2(y). \end{aligned}$$

where $\nu_4 : \mathrm{SL}(2) \to \mathrm{Sp}(4)$ is a 4-dimensional *symplectic* irreducible representation of $\mathrm{SL}(2)$, $\nu_3 : \mathrm{SL}(2) \to \mathrm{SO}(3)$ is a 3-dimensional *orthogonal* irreducible representation of $\mathrm{SL}(2)$ and, as above, $\nu_2 : \mathrm{SL}(2) \to \mathrm{SL}(2)$ is the identity representation. Note that ψ_0 is the Aubert dual of ψ_7, ψ_2 is the Aubert dual of ψ_6, and ψ_4 is the Aubert dual of ψ_5.

These Arthur parameters define the following six Langlands parameters:

$$\begin{aligned} \phi_0(w,x) &= \nu_4(d_w) \oplus \nu_2(d_w), & \phi_7(w,x) &= \nu_4(x) \oplus \nu_2(x), \\ \phi_2(w,x) &= \nu_4(d_w) \oplus \nu_2(x), & \phi_6(w,x) &= \nu_4(x) \oplus \nu_2(d_w), \\ \phi_4(w,x) &= \nu_2(x) \otimes \nu_3(d_w), & \phi_5(w,x) &= \nu_3(x) \otimes \nu_2(d_w). \end{aligned}$$

The remaining two Langlands parameters in $P_\lambda({}^LG)/Z_{\widehat{G}}(\lambda)$ that are not of Arthur type are given here:

$$\phi_1(w,x) = \left(\begin{array}{cc|cc|cc} |w|x_{11} & |w|x_{12} & 0 & 0 & 0 & 0 \\ |w|x_{21} & |w|x_{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & |w|^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & |w|^{-1/2} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & |w|^{-1}x_{11} & |w|^{-1}x_{12} \\ 0 & 0 & 0 & 0 & |w|^{-1}x_{21} & |w|^{-1}x_{22} \end{array}\right),$$

$$\phi_3(w,x) = \left(\begin{array}{cc|cc|cc} |w|^{3/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{11} & 0 & 0 & x_{12} & 0 \\ \hline 0 & 0 & x_{11} & x_{12} & 0 & 0 \\ 0 & 0 & x_{21} & x_{22} & 0 & 0 \\ \hline 0 & -x_{21} & 0 & 0 & -x_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & |w|^{-3/2} \end{array}\right).$$

6.1.2. *L-packets.* In total, there are 15 admissible representations with infinitesimal parameter λ, of which 10 are representations of $G_0(F)$ while 5 are representations of $G_1(F)$. In order to list them, we must enumerate the irreducible representations A_ϕ, for each $\phi \in P_\lambda({}^LG)$. In every case but one, the group A_ϕ is trivial or has order 2; in the latter case, the irreducible representations of these groups are unambiguously labeled with $+$ or $-$; in the former case, we simply elide the trivial representation, such as in the list below.

$$\begin{array}{lcl@{\qquad}lcl}
\Pi_{\phi_0}(G_0(F)) & = & \{\pi(\phi_0)\} & \Pi_{\phi_0}(G_1(F)) & = & \emptyset \\
\Pi_{\phi_1}(G_0(F)) & = & \{\pi(\phi_1)\} & \Pi_{\phi_1}(G_1(F)) & = & \emptyset \\
\Pi_{\phi_2}(G_0(F)) & = & \{\pi(\phi_2,+)\} & \Pi_{\phi_2}(G_1(F)) & = & \{\pi(\phi_2,-)\} \\
\Pi_{\phi_3}(G_0(F)) & = & \{\pi(\phi_3,+),\pi(\phi_3,-)\} & \Pi_{\phi_3}(G_1(F)) & = & \emptyset \\
\Pi_{\phi_4}(G_0(F)) & = & \{\pi(\phi_4,+)\} & \Pi_{\phi_4}(G_1(F)) & = & \{\pi(\phi_4,-)\} \\
\Pi_{\phi_5}(G_0(F)) & = & \{\pi(\phi_5)\} & \Pi_{\phi_5}(G_1(F)) & = & \emptyset \\
\Pi_{\phi_6}(G_0(F)) & = & \{\pi(\phi_6,+)\} & \Pi_{\phi_6}(G_1(F)) & = & \{\pi(\phi_6,-)\} \\
\Pi_{\phi_7}(G_0(F)) & = & \{\pi(\phi_7,++),\pi(\phi_7,--)\} & \Pi_{\phi_7}(G_1(F)) & = & \{\pi(\phi_7,+-),\pi(\phi_7,-+)\}
\end{array}$$

The centraliser of ϕ_7 is the following subgroup of 2-torsion elements $\widehat{T}[2]$ in the diagonal dual torus $\widehat{T}$:

$$Z_{\widehat{G}}(\phi_7) = \left\{ \begin{pmatrix} s_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_1 \end{pmatrix} \in \widehat{T}[2] \mid \quad s_1 = s_2 \right\}.$$

We fix the isomorphism $Z_{\widehat{G}}(\phi_7) \cong \{\pm 1\} \times \{\pm 1\}$ so that the image of $Z(\widehat{G})$ in $Z_{\widehat{G}}(\phi_7)$ is $\{(+1,+1),(-1,-1)\}$; using this isomorphism, we label irreducible representations of $A_{\phi_7} \cong Z_{\widehat{G}}(\phi_7)$ by the symbols $++$, $+-$, $-+$ and $--$. Note that the restriction of these representations to $Z(\widehat{G})$ is trivial for $++$ and $--$ only.

Of these 15 admissible representations, only the representation $\pi(\phi_7,+-)$ of $G_1(F)$ is supercuspidal. In fact, $\pi(\phi_7,+-)$ is a unipotent supercusidal depth-zero representation.

In Lusztig's classification of unipotent representations, $\pi(\phi_7, +-)$ is the case $n = 3$, $a = 1$, $b = 1$ of [13, 7.55]; it corresponds to the unique cuspidal unipotent local system for $\widehat{G}$, see $\tilde{C}_3/(C_3 \times C_0)$ in [13, 7.55]. Lusztig's classification also shows how $\pi(\phi_7, +-)$ may be constructed by compact induction, as follows; see $\tilde{B}_3/(D_1 \times B_2)$ in [13, 7.55]. Let $\underline{G_1}$ be the parahoric $\mathcal{O}_F$-group scheme associated to an almost self-dual lattice chain and the quadratic form at the beginning of Section 6. The generic fibre of $\underline{G_1}$ is the inner form G_1 of G^*, and $\underline{G_1}(\mathcal{O}_F)$ is a maximal parahoric subgroup of the F-points on the generic fibre of $\underline{G_1}$. The reductive quotient $\underline{G_1}^{\mathrm{red}}_{\mathbb{F}_q}$ of the special fibre of $\underline{G_1}$ is $\mathrm{SO}(5) \times \mathrm{SO}(2)$ over $\mathbb{F}_q$, where $\mathrm{SO}(5)$ and $\mathrm{SO}(2)$ are determined, respectively, by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \epsilon & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -\epsilon & 0 \\ 0 & 1 \end{pmatrix},$$

with $\epsilon = \varepsilon \mod \mathcal{O}_F$. Note that the parahoric $\underline{G}_1(\mathcal{O}_K)$ is not hyperspecial. The finite group $\mathrm{SO}(5, \mathbb{F}_q) \times \mathrm{SO}(2, \mathbb{F}_q)$ admits a unique cuspidal unipotent irreducible representation, ${}^o\sigma$. Let $\inf({}^o\sigma)$ be the representation of $\underline{G_1}(\mathcal{O}_F)$ obtained by inflation of ${}^o\sigma$ along $\underline{G_1}(\mathcal{O}_F) \to (\underline{G}_1)^{\mathrm{red}}_{\mathbb{F}_q}(\mathbb{F}_q)$. Now extend $\inf({}^o\sigma)$ to the representation $\inf({}^o\sigma)^+$ of $N_{G_1(F)}(\underline{G_1}(\mathcal{O}_F))$ by tensoring with an unramified character which has order 2 on $N_{G_1(F)}(\underline{G_1}(\mathcal{O}_F))/\underline{G_1}(\mathcal{O}_F)$. Then

$$\pi(\phi_7, +-) = \mathrm{cInd}_{N_{G_1(F)}(\underline{G_1}(\mathcal{O}_F))}^{\underline{G_1}(F)}(\inf(\sigma)^+).$$

We remark that $N_{G_1(F)}(\underline{G_1}(\mathcal{O}_F))$ also admits a smooth model over $\mathcal{O}_F$, for which the reductive quotient of the special fibre is $\mathrm{S}(\mathrm{O}(5) \times \mathrm{O}(2)) \cong \mathrm{SO}(5) \times \mathrm{O}(2)$.

6.1.3. *Multiplicities in standard modules.* In order to describe the other admissible representations appearing in this example, we give the multiplicity of $\pi(\phi, \rho)$ in the standard modules $M(\phi', \rho')$ for representations of the pure form $G_0(F)$. To save space here we write π_i for $\pi(\phi_i)$ and π_i^ϵ for $\pi(\phi_i, \epsilon)$; a similar convention applies to the notation for the standard modules here.

G_0	π_0	π_1	π_2^+	π_3^{++}	π_3^{--}	π_4^+	π_5	π_6^+	π_7^{++}	π_7^{--}
M_0	1	1	1	1	1	2	2	1	1	1
M_1	0	1	0	0	0	1	1	1	1	1
M_2^+	0	0	1	1	0	1	1	1	1	0
M_3^{++}	0	0	0	1	0	0	1	0	1	0
M_3^{--}	0	0	0	0	1	0	1	0	0	1
M_4^+	0	0	0	0	0	1	1	1	1	0
M_5	0	0	0	0	0	0	1	0	1	1
M_6^+	0	0	0	0	0	0	0	1	1	0
M_7^{++}	0	0	0	0	0	0	0	0	1	0
M_7^{--}	0	0	0	0	0	0	0	0	0	1

Let us see show how to calculate row 8 in the table of multiplicities for $G_0(F)$ above. Consider the standard module

$$M_6^+ := \mathrm{Ind}(||^{1/2} \otimes \pi(\nu_4, +))$$

for $G_0(F)$. It is clear that this will contain π_6^+. Moreover, it has an irreducible submodule π_7^{++}. To show there is nothing else, we can compute the Jacquet module of M_6^+ with respect to the standard parabolic subgroup P, whose Levi component is $GL(1) \times SO(5)$. By the geometric lemma, we get

$$\text{s.s. } \mathrm{Jac_P}\, \mathrm{M}_6^+ = ||^{3/2} \otimes \mathrm{Ind}(||^{1/2} \otimes \pi(\nu_2, +)) \oplus ||^{1/2} \otimes \pi(\nu_4, +) \oplus ||^{-1/2} \otimes \pi(\nu_4, +)$$

and

$$\text{s.s. } \mathrm{Ind}(||^{1/2} \otimes \pi(\nu_2, +)) = \pi(\nu_2 \oplus \nu_2, ++) \oplus \pi'$$

where π' is the unique irreducible quotient. Here, s.s. denotes the semi-simplification of the module. On the other hand,

$$\text{s.s. } \mathrm{Jac_P} \pi_6^+ = ||^{-1/2} \otimes \pi(\nu_4, +) \oplus ||^{3/2} \otimes \pi'$$

and

$$\text{s.s. } \mathrm{Jac_P} \pi_7^{++} = ||^{1/2} \otimes \pi(\nu_4, +) \oplus ||^{3/2} \otimes \pi(\nu_2 \oplus \nu_2, ++).$$

Therefore,

$$\text{s.s. } M_6^+ = \pi_6^+ \oplus \pi_7^{++}.$$

This explains row 8 in the table of multiplicities for $G_0(F)$, above.

Passing from G_0 to G_1, we now list the multiplicity of $\pi(\phi, \rho)$ in the standard modules $M(\phi', \rho')$, for representations of the form $G_1(F)$.

G_1	π_2^-	π_4^-	π_6^-	π_7^{-+}	π_7^{+-}
M_2^-	1	1	1	1	0
M_4^-	0	1	1	0	0
M_6^-	0	0	1	1	0
M_7^{-+}	0	0	1	0	0
M_7^{+-}	0	0	0	0	1

6.1.4. *Arthur packets.* In order to describe the component groups A_ψ, consider the torus

$$S := \left\{ \left(\begin{array}{c|cc|cc|c} s_1 & & & & & \\ \hline & s_2 & 0 & & & \\ & 0 & s_3 & & & \\ \hline & & & s_3^{-1} & 0 & \\ & & & 0 & s_2^{-1} & \\ \hline & & & & & s_1^{-1} \end{array} \right) \;\middle|\; s_1 = s_2 \right\} \subset \widehat{T} \subset \widehat{G}.$$

Let $S[2]$ be the 2-torsion subgroup of S; Note that $Z(\widehat{G}) \subset S[2]$. Let us the notation

$$s(s_2, s_3) := \left(\begin{array}{c|cc|cc|c} s_2 & & & & & \\ \hline & s_2 & 0 & & & \\ & 0 & s_3 & & & \\ \hline & & & s_3 & 0 & \\ & & & 0 & s_2 & \\ \hline & & & & & s_2 \end{array} \right) \in S[2]$$

and let $S[2] \cong \{\pm 1\} \times \{\pm 1\}$ be the isomorphism determined by this notation. Then $Z(\widehat{G}) \cong \{\pm 1\}$ is the diagonal subgroup, for which we will use the notation

$$s(s_1, s_1) := \left(\begin{array}{c|cc|cc|c} s_1 & & & & & \\ \hline & s_1 & 0 & & & \\ & 0 & s_1 & & & \\ \hline & & & s_1 & 0 & \\ & & & 0 & s_1 & \\ \hline & & & & & s_1 \end{array}\right) \in Z(\widehat{G}) \subset S[2].$$

We can now give the component groups A_ψ:

$$\begin{array}{rcl@{\qquad\qquad}rcl} A_{\psi_0} & = & S[2], & A_{\psi_7} & = & S[2], \\ A_{\psi_2} & = & S[2], & A_{\psi_6} & = & S[2], \\ A_{\psi_4} & = & Z(\widehat{G}), & A_{\psi_5} & = & Z(\widehat{G}). \end{array}$$

The Arthur packets for admissible representations of $G_0(F)$ with infinitesimal parameter λ are

$$\begin{array}{rcl} \Pi_{\psi_0}(G_0(F)) & = & \{\pi(\phi_0), \pi(\phi_2, +)\}, \\ \Pi_{\psi_2}(G_0(F)) & = & \{\pi(\phi_2, +), \pi(\phi_3, -)\}, \\ \Pi_{\psi_4}(G_0(F)) & = & \{\pi(\phi_4, +)\}, \\ \Pi_{\psi_5}(G_0(F)) & = & \{\pi(\phi_5)\}, \\ \Pi_{\psi_6}(G_0(F)) & = & \{\pi(\phi_6, +), \pi(\phi_7, --)\}, \\ \Pi_{\psi_7}(G_0(F)) & = & \{\pi(\phi_7, ++), \pi(\phi_7, --)\}, \end{array}$$

and the Arthur packets for admissible representations of $G_1(F)$ with infinitesimal parameter λ are

$$\begin{array}{rcl} \Pi_{\psi_0}(G_1(F)) & = & \{\pi(\phi_4, -), \pi(\phi_7(+-)\}, \\ \Pi_{\psi_2}(G_1(F)) & = & \{\pi(\phi_2, -), \pi(\phi_7, +-)\}, \\ \Pi_{\psi_4}(G_1(F)) & = & \{\pi(\phi_4, -), \pi(\phi_7, +-)\}, \\ \Pi_{\psi_5}(G_1(F)) & = & \{\pi(\phi_7, -+), \pi(\phi_7, +-)\}, \\ \Pi_{\psi_6}(G_1(F)) & = & \{\pi(\phi_6, -), \pi(\phi_7, +-)\}, \\ \Pi_{\psi_7}(G_1(F)) & = & \{\pi(\phi_7, -+), \pi(\phi_7, +-)\}. \end{array}$$

For later reference, we arrange these representations into pure Arthur packets.

pure Arthur packets	pure L-packet representations	coronal representations
$\Pi_{\mathrm{pure},\psi_0}(G/F)$	$[\pi(\phi_0), 0]$,	$[\pi(\phi_2, +), 0]$, $[\pi(\phi_4, -), 1]$, $[\pi(\phi_7, +-), 1]$
$\Pi_{\mathrm{pure},\psi_2}(G/F)$	$[\pi(\phi_2, +), 0]$, $[\pi(\phi_2, -), 1]$,	$[\pi(\phi_3, -), 0]$, $[\pi(\phi_7, +-), 1]$
$\Pi_{\mathrm{pure},\psi_4}(G/F)$	$[\pi(\phi_4, +), 0]$, $[\pi(\phi_4, -), 1]$,	$[\pi(\phi_7, +-), 1]$
$\Pi_{\mathrm{pure},\psi_5}(G/F)$	$[\pi(\phi_5), 0]$,	$[\pi(\phi_7, -+), 1]$, $[\pi(\phi_7, +-), 1]$
$\Pi_{\mathrm{pure},\psi_6}(G/F)$	$[\pi(\phi_6, +), 0]$, $[\pi(\phi_6, -), 1]$,	$[\pi(\phi_7, --), 0]$, $[\pi(\phi_7, +-), 1]$
$\Pi_{\mathrm{pure},\psi_7}(G/F)$	$[\pi(\phi_7, ++), 0]$, $[\pi(\phi_7, --), 0]$, $[\pi(\phi_7, -+), 1]$, $[\pi(\phi_7, +-), 1]$	

6.1.5. *Aubert duality.* The following table gives Aubert duality for the admissible representations of $G_0(F)$ with infinitesimal parameter λ.

π	$\hat{\pi}$
$\pi(\phi_0)$	$\pi(\phi_7,++)$
$\pi(\phi_1,+)$	$\pi(\phi_3,+)$
$\pi(\phi_2,+)$	$\pi(\phi_7,--)$
$\pi(\phi_3,+)$	$\pi(\phi_1,+)$
$\pi(\phi_3,-)$	$\pi(\phi_6,+)$
$\pi(\phi_4,+)$	$\pi(\phi_5)$
$\pi(\phi_5)$	$\pi(\phi_4,+)$
$\pi(\phi_6,+)$	$\pi(\phi_3,-)$
$\pi(\phi_7,++)$	$\pi(\phi_0)$
$\pi(\phi_7,--)$	$\pi(\phi_2,+)$

Aubert duality for the admissible representations of $G_1(F)$ with infinitesimal parameter λ is given by the following table.

π	$\hat{\pi}$
$\pi(\phi_2,-)$	$\pi(\phi_6,-)$
$\pi(\phi_4,-)$	$\pi(\phi_7,-+)$
$\pi(\phi_6,-)$	$\pi(\phi_2,-)$
$\pi(\phi_7,+-)$	$\pi(\phi_7,+-)$

The twisting characters χ_{ψ_0}, χ_{ψ_4}, χ_{ψ_5} and χ_{ψ_7} are trivial. The twisting characters χ_{ψ_2} and χ_{ψ_6} are nontrivial, both given $(--)$, using the isomorphisms $A_{\psi_2} = S[2] \cong \{\pm 1\} \times \{\pm 1\}$ and $A_{\psi_6} = S[2] \cong \{\pm 1\} \times \{\pm 1\}$ fixed in Section 6.1.4.

6.1.6. *Stable distributions and endoscopy.* The stable distributions on $G_0(F)$ attached to these Arthur packets are:

$$\begin{aligned}
\Theta^{G_0}_{\psi_0} &= \operatorname{trace}\pi(\phi_0) + \operatorname{trace}\pi(\phi_2,+), & \Theta^{G_0}_{\psi_7} &= \operatorname{trace}\pi(\phi_7,++) + \operatorname{trace}\pi(\phi_7,--),\\
\Theta^{G_0}_{\psi_2} &= \operatorname{trace}\pi(\phi_2,+) - \operatorname{trace}\pi(\phi_3,-), & \Theta^{G_0}_{\psi_6} &= \operatorname{trace}\pi(\phi_6,+) - \operatorname{trace}\pi(\phi_7,--),\\
\Theta^{G_0}_{\psi_4} &= \operatorname{trace}\pi(\phi_4,+), & \Theta^{G_0}_{\psi_5} &= \operatorname{trace}\pi(\phi_5).
\end{aligned}$$

The characters $\langle\,\cdot\,,\pi\rangle_\psi$ of A_ψ are given by

π	$\langle\,\cdot\,,\pi\rangle_{\psi_0}$	$\langle\,\cdot\,,\pi\rangle_{\psi_2}$	$\langle\,\cdot\,,\pi\rangle_{\psi_4}$	$\langle\,\cdot\,,\pi\rangle_{\psi_5}$	$\langle\,\cdot\,,\pi\rangle_{\psi_6}$	$\langle\,\cdot\,,\pi\rangle_{\psi_7}$
$\pi(\phi_0)$	$++$	0	0	0	0	0
$\pi(\phi_2,+)$	$--$	$++$	0	0	0	0
$\pi(\phi_3,-)$	0	$--$	0	0	0	0
$\pi(\phi_4,+)$	0	0	$+$	0	0	0
$\pi(\phi_5)$	0	0	0	$+$	0	0
$\pi(\phi_6,+)$	0	0	0	0	$++$	0
$\pi(\phi_7,++)$	0	0	0	0	0	$++$
$\pi(\phi_7,--)$	0	0	0	0	$--$	$--$

With this, we easily find the coefficients $\langle s s_\psi, \pi\rangle_\psi$ in $\Theta^{G_0}_{\psi,s}$. First calculate $s_\psi := \psi(1,1,-1)$:

$$\begin{aligned}
s_{\psi_0} &= \nu_4(-1)\oplus\nu_2(-1) = s(-1,-1), & s_{\psi_7} &= \nu_4(1)\oplus\nu_2(1) = s(1,1),\\
s_{\psi_2} &= \nu_4(-1)\oplus\nu_2(1) = s(-1,1), & s_{\psi_6} &= \nu_4(1)\oplus\nu_2(-1) = s(1,-1),\\
s_{\psi_4} &= \nu_2(1)\otimes\nu_3(-1) = s(1,1), & s_{\psi_5} &= \nu_3(1)\otimes\nu_2(-1) = s(-1,-1).
\end{aligned}$$

Then, using the notation $s = s(s_2, s_3)$ from Section 6.1.4, we have:

$$\begin{aligned}
\Theta^{G_0}_{\psi_0,s} &= \operatorname{trace}\pi(\phi_0) + s_2 s_3 \ \operatorname{trace}\pi(\phi_2,+),\\
\Theta^{G_0}_{\psi_2,s} &= \operatorname{trace}\pi(\phi_2,+) - s_2 s_3 \ \operatorname{trace}\pi(\phi_3,-),\\
\Theta^{G_0}_{\psi_4,s} &= \operatorname{trace}\pi(\phi_4,+),
\end{aligned}$$

and

$$\begin{aligned}
\Theta^{G_0}_{\psi_7,s} &= \operatorname{trace}\pi(\phi_7,++) + s_2 s_3 \ \operatorname{trace}\pi(\phi_7,--),\\
\Theta^{G_0}_{\psi_6,s} &= \operatorname{trace}\pi(\phi_6,+) - s_2 s_3 \ \operatorname{trace}\pi(\phi_7,--),\\
\Theta^{G_0}_{\psi_5,s} &= \operatorname{trace}\pi(\phi_5).
\end{aligned}$$

We now turn our attention to the distributions on $G_1(F)$ attached to these Arthur packets:

$$\begin{aligned}
\Theta^{G_1}_{\psi_0} &= -\ \operatorname{trace}\pi(\phi_4,+) - \ \operatorname{trace}\pi(\phi_7,+-)\\
\Theta^{G_1}_{\psi_2} &= +\ \operatorname{trace}\pi(\phi_2,-) - \ \operatorname{trace}\pi(\phi_7,+-)\\
\Theta^{G_1}_{\psi_4} &= +\ \operatorname{trace}\pi(\phi_4,-) + \ \operatorname{trace}\pi(\phi_7,+-)
\end{aligned}$$

and

$$\begin{aligned}
\Theta^{G_1}_{\psi_7} &= +\ \operatorname{trace}\pi(\phi_7,-+) + \ \operatorname{trace}\pi(\phi_7,+-)\\
\Theta^{G_1}_{\psi_6} &= +\ \operatorname{trace}\pi(\phi_6,-) - \ \operatorname{trace}\pi(\phi_7,+-)\\
\Theta^{G_1}_{\psi_5} &= -\ \operatorname{trace}\pi(\phi_7,-+) - \ \operatorname{trace}\pi(\phi_7,+-)
\end{aligned}$$

For these representations, the characters $\langle\,\cdot\,,\pi\rangle_\psi$ of A_ψ are given by

π	$\langle\,\cdot\,,\pi\rangle_{\psi_0}$	$\langle\,\cdot\,,\pi\rangle_{\psi_2}$	$\langle\,\cdot\,,\pi\rangle_{\psi_4}$	$\langle\,\cdot\,,\pi\rangle_{\psi_5}$	$\langle\,\cdot\,,\pi\rangle_{\psi_6}$	$\langle\,\cdot\,,\pi\rangle_{\psi_7}$
$\pi(\phi_2,-)$	0	$+-$	0	0	0	0
$\pi(\phi_4,-)$	$-+$	0	$-$	0	0	0
$\pi(\phi_6,-)$	0	0	0	0	$-+$	0
$\pi(\phi_7,-+)$	0	0	0	$-$	0	$-+$
$\pi(\phi_7,+-)$	$+-$	$-+$	$-$	$-$	$+-$	$+-$

With this, we easily find the coefficients $\langle s s_\psi,\pi\rangle_\psi$ in $\Theta^{G_1}_{\psi,s}$, again using the notation $s = s(s_2,s_3)$ or $s = s(s_1,s_1)$ from Section 6.1.4 from which we deduce

$$\begin{aligned}
\Theta^{G_1}_{\psi_0,s} &= -s_2 \operatorname{trace}\pi(\phi_4,+) - s_3 \operatorname{trace}\pi(\phi_7,+-)\\
\Theta^{G_1}_{\psi_2,s} &= +s_3 \operatorname{trace}\pi(\phi_2,-) - s_2 \operatorname{trace}\pi(\phi_7,+-)\\
\Theta^{G_1}_{\psi_4,s} &= +s_1 \operatorname{trace}\pi(\phi_4,-) + s_1 \operatorname{trace}\pi(\phi_7,+-)
\end{aligned}$$

and

$$\begin{aligned}
\Theta^{G_1}_{\psi_7,s} &= +s_2 \operatorname{trace}\pi(\phi_7,-+) + s_3 \operatorname{trace}\pi(\phi_7,+-)\\
\Theta^{G_1}_{\psi_6,s} &= +s_2 \operatorname{trace}\pi(\phi_6,-) - s_3 \operatorname{trace}\pi(\phi_7,+-)\\
\Theta^{G_1}_{\psi_5,s} &= -s_1 \operatorname{trace}\pi(\phi_7,-+) - s_1 \operatorname{trace}\pi(\phi_7,+-)
\end{aligned}$$

The endoscopic group for G attached to $s = s(1,-1)$ or $s = s(-1,1)$ is $G' = \mathrm{SO}(5)\times\mathrm{SO}(3)$, in which case $\Theta^G_{\psi,s}$ is the endoscopic transfer of a stable distribution $\Theta^{G'}_{\psi'}$. We write $\psi' = (\psi^{(2)},\psi^{(1)})$ where $\psi^{(1)}$ is an Arthur parameter for $\mathrm{SO}(3)$ and $\psi^{(2)}$ is an Arthur parameter for $\mathrm{SO}(5)$. The following table gives $\psi^{(1)}$ from Section 2.1.1 and $\psi^{(2)}$ from Section 4.1.1, for each Arthur parameter ψ appearing in Section 6.1.1 that factors

through ${}^LG'$.

§6.1.1 ψ	§4.1.1 $\psi^{(2)}$	§2.1.1 $\psi^{(1)}$
ψ_0	ψ_0	ψ_0
ψ_2	ψ_0	ψ_1
ψ_6	ψ_3	ψ_0
ψ_7	ψ_3	ψ_1

6.2. **Vanishing cycles of perverse sheaves.**

6.2.1. *Vogan variety and its conormal bundle.* The centralizer in $\widehat{G}$ of the infinitesimal parameter $\lambda : W_F \to {}^LG$ is

$$H_\lambda := \left\{ \left(\begin{array}{c|cc|cc|c} h_1 & & & & & \\ \hline & a_2 & b_2 & & & \\ & c_2 & d_2 & & & \\ \hline & & & a_3 & b_3 & \\ & & & c_3 & d_3 & \\ \hline & & & & & h_4 \end{array} \right) \in \widehat{G} \right\} \cong \mathrm{GL}(1) \times \mathrm{GL}(2)$$

We will write $h_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$ and $h_3 = \left(\begin{smallmatrix} a_3 & b_3 \\ c_3 & d_3 \end{smallmatrix}\right)$. Then $h_3 = h_2 \det h_2^{-1}$ and $h_4 = h_1^{-1}$, by the choice of symplectic form J at the beginning of Section 6. The Vogan varieties V_λ and V_λ^* are:

$$V_\lambda = \left\{ \left(\begin{array}{c|cc|cc|c} & u & v & & & \\ \hline & & & z & x & \\ & & & y & -z & \\ \hline & & & & & -v \\ & & & & & u \\ \hline & & & & & \end{array} \right) \right\}, \qquad V_\lambda^* = \left\{ \left(\begin{array}{c|cc|cc|c} & & & & & \\ \hline u' & & & & & \\ v' & & & & & \\ \hline & z' & y' & & & \\ & x' & -z' & & & \\ \hline & & & -v' & u' & \\ \end{array} \right) \right\}$$

so

$$T^*(V_\lambda) = \left\{ \left(\begin{array}{c|cc|cc|c} & u & v & & & \\ \hline u' & & & z & x & \\ v' & & & y & -z & \\ \hline & z' & y' & & & -v \\ & x' & -z' & & & u \\ \hline & & & -v' & u' & \end{array} \right) \;\middle|\; \begin{array}{c} u, v, x, y, z \\ u', v', x', y', z' \end{array} \right\} \subset \widehat{\mathfrak{g}}$$

The action of H_λ on V, V^* and $T^*(V_\lambda)$ is simply the restriction of the adjoint action of $H \subset \widehat{G}$ on $T^*(V) \subset \widehat{\mathfrak{g}}$. This action is given by

$$\begin{aligned} h \cdot \begin{pmatrix} u & v \end{pmatrix} &= h_1 \begin{pmatrix} u & v \end{pmatrix} h_2^{-1} \\ h \cdot \begin{pmatrix} z & x \\ y & -z \end{pmatrix} &= h_2 \begin{pmatrix} z & x \\ y & -z \end{pmatrix} h_3^{-1} \end{aligned}$$

and

$$\begin{aligned} h \cdot \begin{pmatrix} u' \\ v' \end{pmatrix} &= h_2 \begin{pmatrix} u' \\ v' \end{pmatrix} h_1^{-1}, \\ h \cdot \begin{pmatrix} z' & y' \\ x' & -z' \end{pmatrix} &= h_3 \begin{pmatrix} z' & y' \\ x' & -z' \end{pmatrix} h_2^{-1}. \end{aligned}$$

We remark that for $\mu \in \mathbb{C}$,

$$\begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} z & x \\ y & -z \end{pmatrix} = \mu \begin{pmatrix} u & v \end{pmatrix}$$

if and only if

$$h \cdot \begin{pmatrix} u & v \end{pmatrix} \; h \cdot \begin{pmatrix} z & x \\ y & -z \end{pmatrix} = (\mu \det h_2) \; h \cdot \begin{pmatrix} u & v \end{pmatrix}.$$

The H-invariant function $(\,\cdot\,|\,\cdot\,) : T^*(V_\lambda) \to \mathbb{A}^1$ is the quadratic form

$$\left(\begin{array}{c|cc|cc|c} & u & v & & & \\ \hline u' & & & z & x & \\ v' & & & y & -z & \\ \hline & z' & y' & & & -v \\ & x' & -z' & & & u \\ \hline & & & -v' & u' & \end{array}\right) \mapsto 2uu' + 2vv' + xx' + yy' + 2zz'.$$

The H_λ-invariant function $[\cdot,\cdot] : T^*(V_\lambda) \to \mathfrak{h}_\lambda$ is given by

$$\left(\begin{array}{c|cc|cc|c} & u & v & & & \\ \hline u' & & & z & x & \\ v' & & & y & -z & \\ \hline & z' & y' & & & -v \\ & x' & -z' & & & u \\ \hline & & & -v' & u' & \end{array}\right) \mapsto \begin{array}{l} (uu' + vv')H_1 + (xx' + zz')H_2 + (yy' + zz')H_3 \\ +(zy' - xz')E + (yz' - zx')F \end{array}$$

where, $\{H_1, H_2, H_3\}$ is the standard basis for the standard Cartan in $\widehat{\mathfrak{g}}$ and, with reference to $H_\lambda \subset \widehat{G}$ and $\mathfrak{h}_\lambda \subset \widehat{\mathfrak{g}}$, $\{H_1\}$, $\{H_2, H_3, E, F\}$ is the Chevalley basis for $gl(2)$ in $\mathrm{sp}(6)$. Thus, the conormal bundle is

$$T^*_H(V_\lambda) = \left\{ \left(\begin{array}{c|cc|cc|c} & u & v & & & \\ \hline u' & & & z & x & \\ v' & & & y & -z & \\ \hline & z' & y' & & & -v \\ & x' & -z' & & & u \\ \hline & & & -v' & u' & \end{array}\right) \;\middle|\; \begin{array}{c} uu' + vv' = 0 \\ xx' + zz' = 0 \\ yy' + zz' = 0 \\ zy' - xz' = 0 \\ yz' - zx' = 0 \end{array} \right\}$$

Note that the fibre of $(\,\cdot\,|\,\cdot\,) : V_\lambda \times V_\lambda^t \to \mathbb{A}^1$ above 0 properly contains the conormal bundle $T^*_H(V_\lambda)$ as a codimension-4 subvariety.

Although it is possible to continue to work with V and $T^*(V)$ as matrices in $\widehat{\mathfrak{g}}$ and make all the following calculations, we now switch to the perspective on Vogan varieties discussed in Section 0.2.1. This new perspective has several advantages: it is notationally less awkward, it generalises to all classical groups after unramification in the sense of [7, Theorem 3.1.1] and it helps clarify the proper covers which play a crucial role in the calculations of the vanishing cycles that we make later in this section. Write $\widehat{\mathfrak{g}} = \mathrm{sp}(E, J)$, so E is a six-dimensional vector space equipped with the symplectic form described in Section 6.1.1. Let E_1 be the eigenspace of $\lambda(\mathrm{Fr})$ with eigenvalue $q^{3/2}$; let E_2 be the eigenspace of $\lambda(\mathrm{Fr})$ with eigenvalue $q^{1/2}$; let E_3 be the eigenspace of $\lambda(\mathrm{Fr})$ with eigenvalue $q^{-1/2}$; let E_4 be the eigenspace of $\lambda(\mathrm{Fr})$ with eigenvalue $q^{-3/2}$. Then $\mathrm{GL}(E_4) \times \mathrm{GL}(E_3) \times \mathrm{GL}(E_2) \times \mathrm{GL}(E_1)$ acts naturally on the variety

$\mathrm{Hom}(E_3, E_4) \times \mathrm{Hom}(E_2, E_3) \times \mathrm{Hom}(E_1, E_2)$. If we identify E_3 with the dual space E_2^* and E_4 with E_1^* then V may be identified with the subvariety

$$\begin{aligned} V &\cong \left\{ (w_1, w_2, w_3) \in \mathrm{Hom}(E_1, E_2) \times \mathrm{Hom}(E_2, E_2^*) \times \mathrm{Hom}(E_2^*, E_1^*) \mid \begin{array}{l} {}^t w_3 = w_1 \\ {}^t w_2 = w_2 \end{array} \right\} \\ &\cong \left\{ (w, X) \in \mathrm{Hom}(E_1, E_2) \times \mathrm{Hom}(E_2, E_2^*) \mid \ {}^t X = X \ \right\} \\ &\cong \mathrm{Hom}(E_1, E_2) \times \mathrm{Sym}^2(E_2^*) \end{aligned}$$

The action of H on V now corresponds to the natural action of $\mathrm{GL}(E_1) \times \mathrm{GL}(E_2)$ on $\mathrm{Hom}(E_1, E_2) \times \mathrm{Hom}(E_2, E_2^*)$. After choosing bases for E_1 and E_2, the conversion from the matrices in $\widehat{\mathfrak{g}}$ to pairs $(w, X) \in \mathrm{Hom}(E_1, E_2) \times \mathrm{Sym}^2(E_2^*)$ is given by

$$w = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \text{and} \qquad X := \begin{pmatrix} -x & z \\ z & y \end{pmatrix} = \begin{pmatrix} z & x \\ y & -z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We will use coordinates (w, X) for V_λ when convenient. The same perspective gives coordinates (w', X') for V_λ^* where

$$w' = \begin{pmatrix} u' & v' \end{pmatrix} \quad \text{and} \quad X' := \begin{pmatrix} -x' & z' \\ z' & y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z' & y' \\ x' & -z' \end{pmatrix}.$$

In these coordinates, the action of H on V is given by

$$\begin{array}{rclcrcl} h \cdot w &=& {}^t h_2^{-1} w \, {}^t h_1 & \qquad & h \cdot w' &=& h_1 w' h_2^{-1} \\ h \cdot X &=& h_2 X \, {}^t h_2 & & h \cdot X' &=& h_2 X' \, {}^t h_2, \end{array}$$

the H-invariant function $(\,\cdot \mid \cdot\,) : T^*(V_\lambda) \to \mathbb{A}^1$ is given by

$$((w, X) \mid (w', X')) = w'w + \mathrm{trace}\, X'X,$$

and the H-invariant function $[\cdot, \cdot] : T^*(V_\lambda) \to \mathfrak{h}_\lambda$ is given by

$$[(w, X), (w', X')] = (w'w, X'X).$$

In particular, the conormal may be written as

$$T^*(V_\lambda) \cong \{((w, X), (w', X')) \in V \times V^* \mid w'w = 0, X'X = 0\}.$$

6.2.2. *Equivariant local systems and orbit duality.* The variety V_λ is stratified into H-orbits according to the possible values of $\mathrm{rank}\, X$ (either 2, 1 or 0), $\mathrm{rank}\, {}^t w$ (either 1 or 0) and $\mathrm{rank}\, {}^t w X w$ (either 1 or 0). There are eight compatible values for these ranks. We now describe these eight locally closed subvarieties $C \subset V$, the singularities in the closure $\bar{C} \subset V$ and the equivariant local systems on C. For each H-orbit $C \subset V$ except the open orbit $C_7 \subset V$, the H-equivariant fundamental group of C is trivial or of order 2. So in each of these cases we use the notation $\mathbb{1}_C$ for the constant local system and $\mathcal{L}_C$ or $\mathcal{F}_C$ for the non-constant irreducible equivariant local system on C. (The choice of $\mathcal{L}_C$ or $\mathcal{F}_C$ will be explained in Section 6.2.4.)

C_0: Closed orbit:

$$C_0 = \{0\}.$$

This corresponds to the minimal rank values

$$\mathrm{rank}\, X = 0, \qquad \mathrm{rank}\, {}^t w = 0, \qquad \mathrm{rank}\, {}^t w X w = 0.$$

This is the only closed orbit in V_λ.

C_1: Punctured plane:

$$C_1 = \{(w, X) \in V_\lambda \mid X = 0, w \neq 0\}.$$

This corresponds to the rank values

$$\operatorname{rank} X = 0, \qquad \operatorname{rank} {}^t w = 1, \qquad \operatorname{rank} {}^t w X w = 0.$$

While C_1 is not affine, its closure $\bar{C}_1 = \{(w, X) \in V_\lambda \mid X = 0\}$ is $\mathbb{A}^2$. This orbit is not of Arthur type. Since A_{C_1} is trivial, $\mathbb{1}_{C_1}$ is the only simple equivariant local system on C_1.

C_2: Smooth cone:

$$C_2 = \{(w, X) \in V_\lambda \mid \operatorname{rank} X = 1, w = 0\}.$$

This corresponds to the rank values

$$\operatorname{rank} X = 1, \qquad \operatorname{rank} {}^t w = 0, \qquad \operatorname{rank} {}^t w X w = 0.$$

Then C_2 is not an affine variety and the singular locus of its closure

$$\bar{C}_2 \cong \{(x, y, z) \mid xy + z^2 = 0\}$$

is precisely C_0. We remark that $xy + z^2$ is a semi-invariant of V_λ with character $h \mapsto \det h_2^2$. Now $A_{C_2} \cong \{\pm 1\}$; let $\mathcal{F}_{C_2}$ be the equivariant local system for the non-trivial character of A_{C_2}. Then $\mathcal{F}_{C_2}$ coincides with the local system denoted by the same symbol in Section 5.2.3.

C_3: The rank values

$$\operatorname{rank} X = 2, \qquad \operatorname{rank} {}^t w = 0, \qquad \operatorname{rank} {}^t w X w = 0.$$

determine

$$C_3 = \{(w, X) \in V_\lambda \mid \operatorname{rank} X = 2, w = 0\} \cong \{(x, y, z) \mid xy + z^2 \neq 0\}.$$

The closure of C_3 is smooth:

$$\bar{C}_3 = \{(w, X) \in V_\lambda \mid w = 0\} \cong \mathbb{A}^3.$$

This orbit is not of Arthur type. Since $A_{C_3} \cong \{\pm 1\}$, there are two simple equivariant local systems on C_3, denoted by $\mathbb{1}_{C_3}$ and $\mathcal{L}_{C_3}$. Then $\mathcal{L}_{C_3}$ coincides with the local system denoted by the same symbol in Section 5.2.3.

C_4: The rank values

$$\operatorname{rank} X = 1, \qquad \operatorname{rank} {}^t w = 1, \qquad \operatorname{rank} {}^t w X w = 0$$

determine

$$C_4 = \{(w, X) \in V_\lambda \mid \operatorname{rank} X = 1, w \neq 0, Xw = 0\}.$$

The singular locus of the closure

$$\bar{C}_4 \cong \{(u, v, x, y, z) \mid xy + z^2 = 0,\ -xu + zv = 0 = zu + yv\}$$

is C_0. Here, $A_{C_4} \cong \{\pm 1\}$. Let $\mathbb{1}_{C_4}$ and $\mathcal{F}_{C_4}$ be the local systems for the trivial and non-trivial characters, respectively, of A_{C_4}.

C_5: The rank values

$$\operatorname{rank} X = 2, \qquad \operatorname{rank} {}^t w = 1, \qquad \operatorname{rank} {}^t wXw = 0$$

determine

$$C_5 = \{(w, X) \in V_\lambda \mid \operatorname{rank} X = 2, w \neq 0, {}^t wXw = 0\}.$$

The closure of C_5,

$$\bar{C}_5 \cong \{(u, v, x, y, z) \mid -u^2x + 2uvz + v^2y = 0\},$$

has singular locus $\bar{C}_3$. We remark that $-u^2x + 2uvz + v^2y$ is a semi-invariant of V_λ with character $h \mapsto h_1^2$. The group A_{C_5} is trivial.

C_6: The rank values

$$\operatorname{rank} X = 1, \qquad \operatorname{rank} {}^t w = 1, \qquad \operatorname{rank} {}^t wXw = 1$$

determine

$$C_6 = \{(w, X) \in V_\lambda \mid \operatorname{rank} X = 1, w \neq 0, {}^t wXw \neq 0\}.$$

The singular locus of

$$\bar{C}_6 \quad \cong \quad \{(u, v, x, y, z) \mid xy + z^2 = 0\}$$

is $\bar{C}_1$. Then $A_{C_6} \cong \{\pm 1\}$. Let $\mathbb{1}_{C_6}$ and $\mathcal{F}_{C_6}$ be the local systems for the trivial and non-trivial characters, respectively, of A_{C_6}. The local system $\mathcal{F}_{C_6}$ is associated to the double cover from adjoining $d^2 = -u^2x + 2uvz + v^2y$, which is isomorphic to the pullback of the double cover from $\mathcal{F}_{C_2}$.

C_7: Open dense orbit:

$$C_7 = \{(w, X) \in V_\lambda \mid \operatorname{rank} X = 2, w \neq 0, {}^t wXw \neq 0\}.$$

This corresponds to the maximal rank values:

$$\operatorname{rank} X = 2, \qquad \operatorname{rank} {}^t w = 1, \qquad \operatorname{rank} {}^t wXw = 1.$$

Now, $A_{C_7} = S[2] \cong \{\pm 1\} \times \{\pm 1\}$. Let $\mathbb{1}_{C_7}$ be the local system for the trivial character $(++)$ of A_{C_7}; let $\mathcal{L}_{C_7}$ be the local system for the character $(--)$ of A_{C_7}; let $\mathcal{F}_{C_7}$ be the local system for the character $(-+)$ of A_{C_7}; let $\mathcal{E}_{C_7}$ be the local system for the character $(+-)$ of A_{C_7}. Equivalently, $\mathcal{L}_{C_7}$ is the local system on C_7 associated to the double cover $d^2 = xy + z^2$, $\mathcal{F}_{C_7}$ is the local system associated to the double cover $d^2 = -u^2x + 2uvz + v^2y$, and $\mathcal{E}_{C_7}$ is the local system associated to the double cover $d^2 = (xy + z^2)(-u^2x + 2uvz + v^2y)$.

Closure relations for these eight orbits in V, and their dual orbits in V^*, are given as follows:

$$\begin{tikzcd}
 & C_7 = \widehat{C}_0 & & & 5 \\
C_5 = \widehat{C}_4 \arrow[ur] & & C_6 = \widehat{C}_2 \arrow[ul] & & 4 \\
C_3 = \widehat{C}_1 \arrow[u] & & C_4 = \widehat{C}_5 \arrow[u] \arrow[ul] & & 3 \\
C_2 = \widehat{C}_6 \arrow[u] \arrow[urr] & & C_1 = \widehat{C}_3 \arrow[u] & & 2 \\
 & C_0 = \widehat{C}_7 \arrow[ul] \arrow[ur] & & & 0
\end{tikzcd}$$

6.2.3. *Equivariant perverse sheaves.* Table 6.2.3.1 shows the results of calculating $\mathcal{P}|_C$ for every simple equivariant perverse sheaf $\mathcal{IC}(C, \mathcal{L})$ and every stratum C in V, together with the normalised geometric multiplicity matrix, m'_{geo}. Notice that m'_{geo} decomposes into block matrices of size 10×10, 4×4 and 1×1.

We now give a few explicit examples of the technique, sketched in Section 0.2.3, which we used to find the geometric multiplicity matrix.

(a) The calculations from Section 5.2.3 show how to find rows 1–5 and row 11 so here we begin with row 6.

(b) To compute $\mathcal{IC}(\mathbb{1}_{C_4})|_C$ for every H-orbit $C \subset V$, observe that

$$\overline{C}_4 = \left\{(w, X) \in V_\lambda \;\middle|\; {}^t wX = 0,\ \det(X) = 0\right\}.$$

Note that ${}^t wX = 0$ implies $\det(X) = 0$ provided $w \neq 0$. This variety is singular precisely when w and X are both zero; in other words, C_0 is the singular locus of $\overline{C}_4$, as we remarked in Section 6.2.1. The blowup of $\overline{C}_4$ at the origin is:

$$\widetilde{C}_4^{(1)} := \left\{((w, X), [a : b]) \in V_\lambda \times \mathbb{P}^1 \;\middle|\; \begin{matrix} \begin{pmatrix} -b & a \end{pmatrix} w = 0, & \begin{pmatrix} a & b \end{pmatrix} X = 0 \\ {}^t wX = 0, & \det X = 0 \end{matrix} \right\}.$$

Let $\pi^{(1)} : \widetilde{C}_4^{(1)} \to \overline{C}_4$ be the obvious projection. In the definition of $\widetilde{C}_4^{(1)}$, the first two equations imply the second two; this observation greatly simplifies checking the following claims. The cover $\pi^{(1)} : \widetilde{C}_4^{(1)} \to \overline{C}_4$ is proper and the variety $\widetilde{C_4}$ is smooth. Moreover, the fibres of $\pi^{(1)}$ have the following structure:

- above C_4, C_2 and C_1, $\pi^{(1)}$ is an isomorphism;
- the fibre of $\pi^{(1)}$ above C_0 is $\mathbb{P}^1$.

It follows that $\pi^{(1)}$ is semi-small. By the decomposition theorem,

$$\pi^{(1)}_!(\mathbb{1}_{\widetilde{C}_4^{(1)}}[3]) = \mathcal{IC}(\mathbb{1}_{C_4}).$$

By proper base change,

$$\begin{array}{rclcrcl} \mathcal{IC}(\mathbb{1}_{C_4})|_{C_4} & = & \mathbb{1}_{C_4}[3] & \qquad & \mathcal{IC}(\mathbb{1}_{C_4})|_{C_2} & = & \mathbb{1}_{C_2}[3] \\ \mathcal{IC}(\mathbb{1}_{C_4})|_{C_1} & = & \mathbb{1}_{C_1}[3] & & \mathcal{IC}(\mathbb{1}_{C_4})|_{C_0} & = & \mathbb{1}_{C_0}[1] \oplus \mathbb{1}_{C_0}[3], \end{array}$$

and $\mathcal{IC}(\mathbb{1}_{C_4})|_C = 0$ for all other strata C.

TABLE 6.2.3.1. Standard sheaves and perverse sheaves in $\mathsf{Per}_{H_\lambda}(V_\lambda)$, and the normalised geometric multiplicity matrix for $\lambda : W_F \to {}^LG$ introduced at the beginning of Section 6.

$\mathcal{P}$	$\mathcal{P}\vert_{C_0}$	$\mathcal{P}\vert_{C_1}$	$\mathcal{P}\vert_{C_2}$	$\mathcal{P}\vert_{C_3}$	$\mathcal{P}\vert_{C_4}$	$\mathcal{P}\vert_{C_5}$	$\mathcal{P}\vert_{C_6}$	$\mathcal{P}\vert_{C_7}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathbb{1}_{C_0}[0]$	0	0	0	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_1})$	$\mathbb{1}_{C_0}[2]$	$\mathbb{1}_{C_1}[2]$	0	0	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_2})$	$\mathbb{1}_{C_0}[2]$	0	$\mathbb{1}_{C_2}[2]$	0	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_3})$	$\mathbb{1}_{C_0}[3]$	0	$\mathbb{1}_{C_2}[3]$	$\mathbb{1}_{C_3}[3]$	0	0	0	0
$\mathcal{IC}(\mathcal{L}_{C_3})$	$\mathbb{1}_{C_0}[1]$	0	0	$\mathcal{L}_{C_3}[3]$	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_4})$	$\mathbb{1}_{C_0}[1] \oplus \mathbb{1}_{C_0}[3]$	$\mathbb{1}_{C_1}[3]$	$\mathbb{1}_{C_2}[3]$	0	$\mathbb{1}_{C_4}[3]$	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_5})$	$\mathbb{1}_{C_0}[2] \oplus \mathbb{1}_{C_0}[4]$	$\mathbb{1}_{C_1}[4]$	$\mathbb{1}_{C_2}[4]$	$\mathbb{1}_{C_3}[4] \oplus \mathcal{L}_{C_3}[4]$	$\mathbb{1}_{C_4}[4]$	$\mathbb{1}_{C_5}[4]$	0	0
$\mathcal{IC}(\mathbb{1}_{C_6})$	$\mathbb{1}_{C_0}[4]$	$\mathbb{1}_{C_1}[4]$	$\mathbb{1}_{C_2}[4]$	0	$\mathbb{1}_{C_4}[4]$	0	$\mathbb{1}_{C_6}[4]$	0
$\mathcal{IC}(\mathbb{1}_{C_7})$	$\mathbb{1}_{C_0}[5]$	$\mathbb{1}_{C_1}[5]$	$\mathbb{1}_{C_2}[5]$	$\mathbb{1}_{C_3}[5]$	$\mathbb{1}_{C_4}[5]$	$\mathbb{1}_{C_5}[5]$	$\mathbb{1}_{C_6}[5]$	$\mathbb{1}_{C_7}[5]$
$\mathcal{IC}(\mathcal{L}_{C_7})$	$\mathbb{1}_{C_0}[3]$	$\mathbb{1}_{C_1}[3]$	0	$\mathcal{L}_{C_3}[5]$	0	$\mathbb{1}_{C_5}[5]$	0	$\mathcal{L}_{C_7}[5]$
$\mathcal{IC}(\mathcal{F}_{C_2})$	0	0	$\mathcal{F}_{C_2}[2]$	0	0	0	0	0
$\mathcal{IC}(\mathcal{F}_{C_4})$	0	0	$\mathcal{F}_{C_2}[3]$	0	$\mathcal{F}_{C_4}[3]$	0	0	0
$\mathcal{IC}(\mathcal{F}_{C_6})$	0	0	$\mathcal{F}_{C_2}[4]$	0	$\mathcal{F}_{C_4}[4]$	0	$\mathcal{F}_{C_6}[4]$	0
$\mathcal{IC}(\mathcal{F}_{C_7})$	0	0	$\mathcal{F}_{C_2}[5]$	0	0	0	$\mathcal{F}_{C_6}[5]$	$\mathcal{F}_{C_7}[5]$
$\mathcal{IC}(\mathcal{E}_{C_7})$	0	0	0	0	0	0	0	$\mathcal{E}_{C_7}[5]$

	$\mathbb{1}^\natural_{C_0}$	$\mathbb{1}^\natural_{C_1}$	$\mathbb{1}^\natural_{C_2}$	$\mathbb{1}^\natural_{C_3}$	$\mathcal{L}^\natural_{C_3}$	$\mathbb{1}^\natural_{C_4}$	$\mathbb{1}^\natural_{C_5}$	$\mathbb{1}^\natural_{C_6}$	$\mathbb{1}^\natural_{C_7}$	$\mathcal{L}^\natural_{C_7}$	$\mathcal{F}^\natural_{C_2}$	$\mathcal{F}^\natural_{C_4}$	$\mathcal{F}^\natural_{C_6}$	$\mathcal{F}^\natural_{C_7}$	$\mathcal{E}^\natural_{C_7}$
$\mathbb{1}^\sharp_{C_0}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbb{1}^\sharp_{C_1}$	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbb{1}^\sharp_{C_2}$	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbb{1}^\sharp_{C_3}$	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{L}^\sharp_{C_3}$	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$\mathbb{1}^\sharp_{C_4}$	2	1	1	0	0	1	0	0	0	0	0	0	0	0	0
$\mathbb{1}^\sharp_{C_5}$	2	1	1	1	1	1	1	0	0	0	0	0	0	0	0
$\mathbb{1}^\sharp_{C_6}$	1	1	1	0	0	1	0	1	0	0	0	0	0	0	0
$\mathbb{1}^\sharp_{C_7}$	1	1	1	1	0	1	1	1	1	0	0	0	0	0	0
$\mathcal{L}^\sharp_{C_7}$	1	1	0	0	1	0	1	0	0	1	0	0	0	0	0
$\mathcal{F}^\sharp_{C_2}$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$\mathcal{F}^\sharp_{C_4}$	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
$\mathcal{F}^\sharp_{C_6}$	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0
$\mathcal{F}^\sharp_{C_7}$	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0
$\mathcal{E}^\sharp_{C_7}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

(c) Next, we show how to compute $\mathcal{IC}(\mathcal{F}_{C_4})$. The singular variety $\overline{C}_4$ also admits a finite double cover:

$$\widetilde{C}_4^{(2)} := \left\{ ((w,X),(\alpha,\beta)) \in V_\lambda \times \mathbb{A}^2 \;\middle|\; \begin{array}{cc} X = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\begin{pmatrix}\alpha & \beta\end{pmatrix}, & \begin{pmatrix}\alpha & \beta\end{pmatrix} w = 0 \\ {}^t wX = 0, & \det X = 0 \end{array} \right\}.$$

Again, the first two equations imply the second two. This variety is singular precisely when w, X, and (α,β) are all zero. Consider the pullback:

$$\begin{tikzcd} & \widetilde{C}_4^{(3)} \arrow[dl] \arrow[dr] \arrow[dd, "\pi^{(3)}"] & \\ \widetilde{C}_4^{(1)} \arrow[dr, "\pi^{(1)}"'] & & \widetilde{C}_4^{(2)} \arrow[dl, "\pi^{(2)}"] \\ & \overline{C}_4. & \end{tikzcd}$$

Then $\widetilde{C}_4^{(3)}$ is smooth and the projections onto $\widetilde{C}_4^{(2)}$, $\widetilde{C}_4^{(1)}$ and $\overline{C}_4$ are all proper. The fibres of $\pi^{(3)} : \widetilde{C}_4^{(3)} \to \overline{C}_4$ have the following structure:

- the fibre of $\pi^{(3)}$ over C_4 is the non-split double cover of C_4;
- the fibre of $\pi^{(3)}$ over C_2 is the non-split double cover of C_2;
- the fibre of $\pi^{(3)}$ over C_1 is isomorphic to C_1;
- the fibre of $\pi^{(3)}$ over C_0 is $\mathbb{P}^1$.

It follows that $\pi^{(3)}$ is semi-small and, by the Decomposition Theorem, that:

$$\pi^{(3)}_!(\mathbb{1}_{\widetilde{C}_4^{(3)}}[3]) = \mathcal{IC}(\mathbb{1}_{C_4}) \oplus \mathcal{IC}(\mathcal{F}_{C_4}).$$

It now follows that:

$$\begin{array}{rclrcl} \mathcal{IC}(\mathcal{F}_{C_4})|_{C_4} & = & \mathcal{F}_{C_4}[3] & \mathcal{IC}(\mathcal{F}_{C_4})|_{C_2} & = & \mathcal{F}_{C_2}[3] \\ \mathcal{IC}(\mathcal{F}_{C_4})|_{C_1} & = & 0 & \mathcal{IC}(\mathcal{F}_{C_4})|_{C_0} & = & 0. \end{array}$$

We simply list the other covers needed to calculate $\mathcal{P}|_C$ in all other cases except $\mathcal{P} = \mathcal{E}_7$, following the procedure illustrated above in the cases $\mathcal{P} = \mathcal{IC}(\mathbb{1}_{C_4})$ and $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_4})$.

- $\widetilde{C}_5 = \left\{ ((w,X),[a:b]) \in V \times \mathbb{P}^1 \mid \begin{pmatrix}a & b\end{pmatrix} X \begin{pmatrix}a\\ b\end{pmatrix} = 0,\ \begin{pmatrix}-b & a\end{pmatrix} w = 0 \right\}$
- $\widetilde{C}_6^{(1)} = \left\{ ((w,X),[a:b]) \in V \times \mathbb{P}^1 \mid \begin{pmatrix}a & b\end{pmatrix} X = 0 \right\}$
- $\widetilde{C}_6^{(2)} = \left\{ ((w,X),(\alpha,\beta)) \in V \times \mathbb{A}^2 \mid X = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\begin{pmatrix}\alpha & \beta\end{pmatrix} \right\}$
- $\widetilde{C}_7^{(1)} = \left\{ (w,X,[a:b]) \in V \times \mathbb{P}^1 \mid \begin{pmatrix}a & b\end{pmatrix} X \begin{pmatrix}a\\ b\end{pmatrix} = 0 \right\}$
- $\widetilde{C}_7^{(2)} = \left\{ ((w,X),[a:b:r]) \in V \times \mathbb{P}^2 \mid \begin{pmatrix}a & b\end{pmatrix} X \begin{pmatrix}a\\ b\end{pmatrix} = r^2,\ \begin{pmatrix}-b & a\end{pmatrix} w = 0 \right\}$

Finally there is the most complex example: the smooth cover $\widetilde{V}$ of $\overline{C_7} = V$ needed to understand $\mathcal{IC}(\mathcal{E}_7)$. The construction of the smooth cover $\widetilde{V}$ of V proceeds by first adjoining a square root of

$$(-u^2x + 2uvz + v^2y)(xy + z^2).$$

This results in a variety which is singular on $\overline{C}_4$. After blowing up along $\overline{C}_4$ the result will still be singular along $\overline{C_3}$, so a further blow up along $\overline{C_3}$ is needed. The following steps construct $\widetilde{V}$ in detail.

(i) Let $\widetilde{V}^{(1)}$ be the blow up V along $\overline{C}_4$ This equivalent to adding coordinates $[a : b] \in \mathbb{P}^1$ and the condition

$$\begin{pmatrix} a & b \end{pmatrix} Xw = 0,$$

because the two equations $Xw = 0$ define $\overline{C}_4$.

(ii) Let $\widetilde{V}^{(2)}$ be the blow up of $\widetilde{C}_7^{(1)}$ along $\overline{C}_3$. For this one must add coordinates $[c : d] \in \mathbb{P}^1$ with the condition

$$\begin{pmatrix} -d & c \end{pmatrix} w = 0,$$

because the equation $w = 0$ defines $\overline{C}_3$. The additional equation necessary to define the blow up is

$$\begin{pmatrix} a & b \end{pmatrix} X \begin{pmatrix} c \\ d \end{pmatrix} = 0.$$

(iii) Next, we replace $[a : b]$ with $[a : b : r]$ and add the equation

$$\begin{pmatrix} a & b \end{pmatrix} X \begin{pmatrix} a \\ b \end{pmatrix} = r^2.$$

The resulting variety, $\widetilde{V}^{(3)}$ has coordinates:

$$(w, X, [a : b : r], [c : d])$$

together with all the above equations. Then $\widetilde{V}^{(3)}$ is a double cover of $\widetilde{V}^{(2)}$ and is singular precisely when

$$X \begin{pmatrix} a \\ b \end{pmatrix} = 0 \qquad \text{and} \qquad [a : b] = [c : d].$$

(iv) We now form the blowup $\widetilde{V}$ of $\widetilde{V}^{(3)}$ along the singular locus. In order to have homogeneous equations we write our relations in the form

$$X \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix} = 0 \qquad \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} d \\ -c \end{pmatrix} = 0.$$

Then $\widetilde{V}$ is formed by introducing coordinates $[Y : y]$, where Y is a 2 by 2 matrix, and the conditions

$$X \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix} y = Y \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} d \\ -c \end{pmatrix}$$

and

$$\begin{pmatrix} c \\ d \end{pmatrix} Y = 0 \qquad \mathrm{Trace}(Y) = 0.$$

Note that $[c : d]$ determines Y up to rescaling.

6.2.4. *Cuspidal support decomposition and Fourier transform.* Up to conjugation, $\widehat{G} = \mathrm{Sp}(6)$ admits three cuspidal Levi subgroups: $\widehat{G} = \mathrm{Sp}(6)$ itself, $\widehat{M} = \mathrm{Sp}(2)\times\mathrm{GL}(1)\times\mathrm{GL}(1)$ and $\widehat{T} = \mathrm{GL}(1) \times \mathrm{GL}(1) \times \mathrm{GL}(1)$. Simple objects in these three subcategories are listed below. This decomposition is responsible for the choice of symbols $\mathcal{L}$, $\mathcal{F}$ and $\mathcal{E}$ made in Section 6.2.3.

$\mathsf{Per}_H(V)_{\check{T}}$		$\mathsf{Per}_H(V)_{\check{M}}$	$\mathsf{Per}_H(V)_{\check{G}}$
$\mathcal{IC}(\mathbb{1}_{C_0})$			
$\mathcal{IC}(\mathbb{1}_{C_1})$			
$\mathcal{IC}(\mathbb{1}_{C_2})$		$\mathcal{IC}(\mathcal{F}_{C_2})$	
$\mathcal{IC}(\mathbb{1}_{C_3})$	$\mathcal{IC}(\mathcal{L}_{C_3})$		
$\mathcal{IC}(\mathbb{1}_{C_4})$		$\mathcal{IC}(\mathcal{F}_{C_4})$	
$\mathcal{IC}(\mathbb{1}_{C_5})$			
$\mathcal{IC}(\mathbb{1}_{C_6})$		$\mathcal{IC}(\mathcal{F}_{C_6})$	
$\mathcal{IC}(\mathbb{1}_{C_7})$	$\mathcal{IC}(\mathcal{L}_{C_7})$	$\mathcal{IC}(\mathcal{F}_{C_7})$	$\mathcal{IC}(\mathcal{E}_{C_7})$

The Fourier transform respects the cuspidal support decomposition:

$$
\begin{array}{rcl}
\mathsf{Ft} : \mathsf{Per}_{H_\lambda}(V_\lambda) & \longrightarrow & \mathsf{Per}_{H_\lambda}(V_\lambda^*) \\
\mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_0^*}) = \mathcal{IC}(\mathbb{1}_{C_7^t}) \\
\mathcal{IC}(\mathbb{1}_{C_1}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_1^*}) = \mathcal{IC}(\mathbb{1}_{C_3^t}) \\
\mathcal{IC}(\mathbb{1}_{C_2}) & \mapsto & \mathcal{IC}(\mathcal{L}_{C_0^*}) = \mathcal{IC}(\mathcal{L}_{C_7^t}) \\
\mathcal{IC}(\mathbb{1}_{C_3}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_3^*}) = \mathcal{IC}(\mathbb{1}_{C_1^t}) \\
\mathcal{IC}(\mathcal{L}_{C_3}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_2^*}) = \mathcal{IC}(\mathbb{1}_{C_6^t}) \\
\mathcal{IC}(\mathbb{1}_{C_4}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_4^*}) = \mathcal{IC}(\mathbb{1}_{C_5^t}) \\
\mathcal{IC}(\mathbb{1}_{C_5}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_5^*}) = \mathcal{IC}(\mathbb{1}_{C_4^t}) \\
\mathcal{IC}(\mathbb{1}_{C_6}) & \mapsto & \mathcal{IC}(\mathcal{L}_{C_1^*}) = \mathcal{IC}(\mathcal{L}_{C_3^t}) \\
\mathcal{IC}(\mathbb{1}_{C_7}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_7^*}) = \mathcal{IC}(\mathbb{1}_{C_0^t}) \\
\mathcal{IC}(\mathcal{L}_{C_7}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_6^*}) = \mathcal{IC}(\mathbb{1}_{C_2^t}) \\
\mathcal{IC}(\mathcal{F}_{C_2}) & \mapsto & \mathcal{IC}(\mathcal{F}_{C_2^*}) = \mathcal{IC}(\mathcal{F}_{C_6^t}) \\
\mathcal{IC}(\mathcal{F}_{C_4}) & \mapsto & \mathcal{IC}(\mathcal{F}_{C_0^*}) = \mathcal{IC}(\mathcal{F}_{C_7^t}) \\
\mathcal{IC}(\mathcal{F}_{C_6}) & \mapsto & \mathcal{IC}(\mathcal{F}_{C_6^*}) = \mathcal{IC}(\mathcal{F}_{C_2^t}) \\
\mathcal{IC}(\mathcal{F}_{C_7}) & \mapsto & \mathcal{IC}(\mathcal{F}_{C_5^*}) = \mathcal{IC}(\mathcal{F}_{C_4^t}) \\
\mathcal{IC}(\mathcal{E}_{C_7}) & \mapsto & \mathcal{IC}(\mathcal{E}_{C_0^*}) = \mathcal{IC}(\mathcal{E}_{C_7^t})
\end{array}
$$

6.2.5. *Equivariant perverse sheaves on the regular conormal bundle.* For each stratum C, we pick $(x,\xi) \in T^*_C(V)_{\mathrm{reg}}$ such that the H-orbit $T^*_C(V)_{\mathrm{sreg}}$ of (x,ξ) is open in $T^*_{C_i}(V)_{\mathrm{reg}}$. Then, we find all equivariant local systems on each $T^*_C(V)_{\mathrm{sreg}}$. The perverse extensions of these local systems to the regular conormal bundle $T^*_H(V)_{\mathrm{reg}}$ will be needed when we compute vanishing cycles of perverse sheave on V in Section 6.2.6. Here we revert to expressing V as a subvariety in $\widehat{\mathfrak{g}}$, largely for typographic reasons.

C_0: Base point for $T^*_{C_0}(V_\lambda)_{\mathrm{sreg}}$:

$$
(x_0,\xi_0) = \left(\begin{array}{c|cc|cc|c}
 & 0 & 0 & & & \\
\hline
1 & & & 0 & 0 & \\
0 & & & 0 & 0 & \\
\hline
 & 0 & 1 & & & 0 \\
 & 1 & 0 & & & 0 \\
\hline
 & & & 0 & 1 &
\end{array}\right)
$$

Equivariant fundamental group $A_{(x_0,\xi_0)}$ is $Z_{H_\lambda}((x_0,\xi_0)) = S[2]$. Thus, $T^*_{C_0}(V_\lambda)_{\mathrm{sreg}}$ carries four local systems. The following table displays how we label equivariant local systems on $T^*_{C_0}(V_\lambda)_{\mathrm{sreg}}$ by showing the matching representation of $A_{(x_0,\xi_0)}$:

$\mathsf{Loc}_{H_\lambda}(T^*_{C_0}(V_\lambda)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_0}$	$\mathcal{L}_{\mathcal{O}_0}$	$\mathcal{F}_{\mathcal{O}_0}$	$\mathcal{E}_{\mathcal{O}_0}$
$\mathsf{Rep}(A_{(x_0,\xi_0)})$:	$++$	$--$	$-+$	$+-$

The map on equivariant fundamental groups $A_{(x_0,\xi_0)} \to A_{x_0}$ induced from the projection $T^*_{C_0}(V)_{\mathrm{sreg}} \to C_0$ is trivial; on the other hand, the map on equivariant fundamental groups $A_{(x_0,\xi_0)} \to A_{\xi_0}$ induced from the projection $T^*_{C_0}(V)_{\mathrm{sreg}} \to C_0^* = C_7^t$ is the identity isomorphism.

$$\begin{tikzcd} & S[2] \arrow[d, "\mathrm{id}"] & \\ 1 = A_{x_0} & A_{(x_0,\xi_0)} \arrow[l] \arrow[r, "\mathrm{id}"] & A_{\xi_0} \end{tikzcd}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \mathsf{Per}_H(C_0) & \to & \mathsf{Per}_H(T^*_{C_0}(V)_{\mathrm{reg}}) \\ \mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) \\ & & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}) \\ & & \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0}) \\ & & \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}) \end{array}$$

C_1: Base point for $T^*_{C_1}(V_\lambda)_{\mathrm{sreg}}$:

$$(x_1,\xi_1) = \left(\begin{array}{c|cc|cc|c} & 1 & 0 & & & \\ \hline 0 & & & 0 & 0 & \\ 0 & & & 0 & 0 & \\ \hline & 0 & 1 & & & 0 \\ & 1 & 0 & & & 1 \\ \hline & & & 0 & 0 & \end{array}\right),$$

Equivariant fundamental group $A_{(x_1,\xi_1)}$ of $T^*_{C_1}(V_\lambda)_{\mathrm{sreg}}$ is $Z_{H_\lambda}((x_1,\xi_1)) = S[2]$. Thus, $T^*_{C_1}(V_\lambda)_{\mathrm{reg}}$ carries four local systems. The following table displays how we label equivariant local systems on $T^*_{C_1}(V_\lambda)_{\mathrm{sreg}}$ by showing the matching representation of $A_{(x_1,\xi_1)}$:

$\mathsf{Loc}_{H_\lambda}(T^*_{C_1}(V_\lambda)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_1}$	$\mathcal{L}_{\mathcal{O}_1}$	$\mathcal{F}_{\mathcal{O}_1}$	$\mathcal{E}_{\mathcal{O}_1}$
$\mathsf{Rep}(A_{(x_1,\xi_1)})$:	$++$	$--$	$-+$	$+-$

For use below, we remark that $\mathcal{L}_{\mathcal{O}_1}$ is the local system associated to the double cover arising from taking $\sqrt{\det X'}$.

The map on equivariant fundamental groups $A_{(x_1,\xi_1)} \to A_{x_1}$ induced from the projection $T^*_{C_1}(V)_{\mathrm{sreg}} \to C_1$ is trivial; on the other hand, the map on equivariant fundamental groups $A_{(x_1,\xi_1)} \to A_{\xi_1}$ induced from the projection $T^*_{C_1}(V)_{\mathrm{sreg}} \to C_1^* = C_3^t$ is $(s_2,s_3) \mapsto s_2s_3$.

$$\begin{tikzcd} & S[2] \arrow[d, "\mathrm{id}"] & \\ 1 = A_{x_1} & A_{(x_1,\xi_1)} \arrow[l] \arrow[r, "(s_2,s_3)\mapsto s_2s_3"] & A_{\xi_1} = \{\pm 1\} \end{tikzcd}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \mathsf{Loc}_H(C_1) & \to & \mathsf{Loc}_H(T^*_{C_1}(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C_1} & \mapsto & \mathbb{1}_{\mathcal{O}_1} \\ & & \mathcal{L}_{\mathcal{O}_1} \\ & & \mathcal{F}_{\mathcal{O}_1} \\ & & \mathcal{E}_{\mathcal{O}_1} \end{array}$$

C_2: Base point for $T^*_{C_2}(V)_{\mathrm{sreg}}$:

$$(x_2,\xi_2) = \left(\begin{array}{c|cc|cc|c} & 0 & 0 & & & \\ \hline 1 & & & 0 & 0 & \\ 0 & & & 1 & 0 & \\ \hline & 0 & 0 & & & 0 \\ & 1 & 0 & & & 0 \\ \hline & & & 0 & 1 & \end{array}\right)$$

The equivariant fundamental group for $T^*_{C_2}(V)_{\mathrm{sreg}}$ is $A_{(x_2,\xi_2)} = Z_{H_\lambda}((x_2,\xi_2)) = S[2]$. Thus, $T^*_{C_2}(V_\lambda)_{\mathrm{reg}}$ carries four local systems.

$\mathsf{Loc}_{H_\lambda}(T^*_{C_2}(V_\lambda)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_2}$	$\mathcal{L}_{\mathcal{O}_2}$	$\mathcal{F}_{\mathcal{O}_2}$	$\mathcal{E}_{\mathcal{O}_2}$
$\mathsf{Rep}(A_{(x_2,\xi_2)})$:	$++$	$--$	$-+$	$+-$

The map on equivariant fundamental groups $A_{(x_2,\xi_2)} \to A_{x_2}$ induced from the projection $T^*_{C_2}(V)_{\mathrm{sreg}} \to C_2$ is given by projection to the second factor while the map on equivariant fundamental groups $A_{(x_2,\xi_2)} \to A_{\xi_2}$ induced from the projection $T^*_{C_2}(V)_{\mathrm{sreg}} \to C_2^* = C_6^t$ is projection to the first factor:

$$\begin{tikzcd} & S[2] \arrow[d, "\mathrm{id}"] & \\ \{\pm 1\} = A_{x_2} & A_{(x_2,\xi_2)} \arrow[l, "s_3 \leftarrow\!\shortmid (s_2,s_3)"'] \arrow[r, "(s_2,s_3)\mapsto s_2"] & A_{\xi_2} = \{\pm 1\} \end{tikzcd}$$

Pull-backalong the bundle map:

$$\begin{array}{ccc} \mathsf{Loc}_H(C_2) & \to & \mathsf{Loc}_H(T^*_{C_2}(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C_2} & \mapsto & \mathbb{1}_{\mathcal{O}_2} \\ & & \mathcal{L}_{\mathcal{O}_2} \\ & & \mathcal{F}_{\mathcal{O}_2} \\ \mathcal{F}_{C_2} & \mapsto & \mathcal{E}_{\mathcal{O}_2} \end{array}$$

C_3: Base point for $T^*_{C_3}(V)_{\mathrm{sreg}}$:

$$(x_3,\xi_3) = \left(\begin{array}{c|cc|cc|c} & 0 & 0 & & & \\ \hline 1 & & & 0 & 1 & \\ 0 & & & 1 & 0 & \\ \hline & 0 & 0 & & & 0 \\ & 0 & 0 & & & 0 \\ \hline & & & 0 & 1 & \end{array}\right)$$

The equivariant fundamental group for $T^*_{C_3}(V)_{\mathrm{sreg}}$ is $A_{(x_3,\xi_3)} = Z_{H_\lambda}((x_3,\xi_3)) = S[2]$. Thus, $T^*_{C_3}(V_\lambda)_{\mathrm{reg}}$ carries four local systems.

$\mathsf{Loc}_{H_\lambda}(T^*_{C_3}(V_\lambda)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_3}$	$\mathcal{L}_{\mathcal{O}_3}$	$\mathcal{F}_{\mathcal{O}_3}$	$\mathcal{E}_{\mathcal{O}_3}$
$\mathsf{Rep}(A_{(x_3,\xi_3)})$:	$++$	$--$	$-+$	$+-$

The map on equivariant fundamental groups $A_{(x_3,\xi_3)} \to A_{x_3}$ induced from the projection $T^*_{C_3}(V)_{\mathrm{sreg}} \to C_2$ has kernel $Z(H)$, while $A_{(x_3,\xi_3)} \to A_{\xi_3}$ is trivial.

$$\begin{tikzcd} & S[2] \arrow[d, "\mathrm{id}"] & \\ \{\pm 1\} = A_{x_3} & A_{(x_3,\xi_3)} \arrow[l, "s_2 s_3 \leftarrow\!\shortmid (s_2,s_3)"'] \arrow[r] & A_{\xi_3} = 1 \end{tikzcd}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \mathsf{Loc}_H(C_3) & \to & \mathsf{Loc}_H(T^*_{C_3}(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C_3} & \mapsto & \mathbb{1}_{\mathcal{O}_3} \\ \mathcal{L}_{C_3} & \mapsto & \mathcal{L}_{\mathcal{O}_0} \\ & & \mathcal{F}_{\mathcal{O}_0} \\ & & \mathcal{E}_{\mathcal{O}_0} \end{array}$$

C_4: Base point for $T^*_{C_4}(V_\lambda)_{\mathrm{sreg}}$:

$$(x_4,\xi_4) = \left(\begin{array}{c|cc|cc|c} & 1 & 0 & & & \\ \hline 0 & & & 0 & 0 & \\ 1 & & & 1 & 0 & \\ \hline & 1 & 0 & & & 0 \\ & 0 & -1 & & & 1 \\ \hline & & & -1 & 0 & \end{array}\right)$$

The equivariant fundamental group of $T^*_{C_4}(V_\lambda)_{\mathrm{sreg}}$ is $A_{(x_4,\xi_4)} = Z_{H_\lambda}((x_4,\xi_4)) = Z(\widehat{G})$. Thus, $T^*_{C_4}(V_\lambda)_{\mathrm{reg}}$ carries two local systems.

$\mathsf{Loc}_{H_\lambda}(T^*_{C_4}(V_\lambda)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_4}$	$\mathcal{F}_{\mathcal{O}_4}$
$\mathsf{Rep}(A_{(x_4,\xi_4)})$:	$+$	$-$

The map on equivariant fundamental groups $A_{(x_4,\xi_4)} \to A_{x_4}$ induced from the projection $T^*_{C_4}(V)_{\mathrm{sreg}} \to C_4$ is the identity isomorphism, while $A_{(x_4,\xi_4)} \to A_{\xi_4}$ is trivial.

$$\begin{tikzcd} & Z(\widehat{G}) \arrow[d, "\mathrm{id}"] & \\ \{\pm 1\} = A_{x_4} & A_{(x_4,\xi_4)} \arrow[l, "\mathrm{id}"'] \arrow[r] & A_{\xi_4} = 1 \end{tikzcd}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \mathsf{Loc}_H(C_4) & \to & \mathsf{Loc}_H(T^*_{C_4}(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C_4} & \mapsto & \mathbb{1}_{\mathcal{O}_4} \\ \mathcal{F}_{C_4} & \mapsto & \mathcal{F}_{\mathcal{O}_4} \end{array}$$

C_5: Base point for $T^*_{C_5}(V_\lambda)_{\mathrm{sreg}}$:

$$(x_5,\xi_5) = \left(\begin{array}{c|cc|cc|c} & 0 & 1 & & & \\ \hline 1 & & & 1 & 0 & \\ 0 & & & 0 & -1 & \\ \hline & 0 & 1 & & & -1 \\ & 0 & 0 & & & 0 \\ \hline & & & 0 & 1 & \end{array}\right)$$

The equivariant fundamental group of $T^*_{C_5}(V_\lambda)_{\mathrm{sreg}}$ is $A_{(x_5,\xi_5)} = Z_{H_\lambda}((x_5,\xi_5)) = Z(\widehat{G})$. Thus, $T^*_{C_5}(V_\lambda)_{\mathrm{reg}}$ carries two local systems.

$\mathsf{Loc}_{H_\lambda}(T^*_{C_5}(V_\lambda)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_5}$	$\mathcal{F}_{\mathcal{O}_5}$
$\mathsf{Rep}(A_{(x_5,\xi_5)})$:	$+$	$-$

The map on equivariant fundamental groups $A_{(x_5,\xi_5)} \to A_{x_5}$ induced from the projection $T^*_{C_5}(V)_{\mathrm{sreg}} \to C_5$ is trivial, while $A_{(x_5,\xi_5)} \to A_{\xi_5}$ is the identity isomorphism.

$$\begin{array}{ccccc} & & Z(\widehat{G}) & & \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 = A_{x_5} & \longleftarrow & A_{(x_5,\xi_5)} & \xrightarrow{\mathrm{id}} & A_{\xi_5} = \{\pm\} \end{array}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \mathsf{Loc}_H(C_5) & \to & \mathsf{Loc}_H(T^*_{C_5}(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C_5} & \mapsto & \mathbb{1}_{\mathcal{O}_5} \\ & & \mathcal{F}_{\mathcal{O}_5} \end{array}$$

C_6: Base point for $T^*_{C_6}(V_\lambda)_{\mathrm{sreg}}$:

$$(x_6,\xi_6) = \left(\begin{array}{c|cc|cc|c} & 1 & 0 & & & \\ \hline 0 & & & 0 & 1 & \\ 0 & & & 0 & 0 & \\ \hline & 0 & 1 & & & 0 \\ & 0 & 0 & & & 1 \\ \hline & & & 0 & 0 & \end{array}\right)$$

The equivariant fundamental group of $T^*_{C_6}(V_\lambda)_{\mathrm{sreg}}$ is $A_{(x_6,\xi_6)} = Z_{H_\lambda}((x_6,\xi_6)) = S[2]$. Thus, $T^*_{C_6}(V_\lambda)_{\mathrm{reg}}$ carries four local systems.

$\mathsf{Loc}_{H_\lambda}(T^*_{C_6}(V_\lambda)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_6}$	$\mathcal{L}_{\mathcal{O}_6}$	$\mathcal{F}_{\mathcal{O}_6}$	$\mathcal{E}_{\mathcal{O}_6}$
$\mathsf{Rep}(A_{(x_6,\xi_6)})$:	$++$	$--$	$-+$	$+-$

The map on equivariant fundamental groups $A_{(x_6,\xi_6)} \to A_{x_6}$ induced from the projection $T^*_{C_6}(V)_{\mathrm{sreg}} \to C_6$ is given by projection to the first factor while the map on equivariant fundamental groups $A_{(x_6,\xi_6)} \to A_{\xi_6}$ induced from the projection

$T^*_{C_6}(V)_{\mathrm{sreg}} \to C_6^* = C_2^t$ is projection to the second factor:

$$\begin{tikzcd} & S[2] \arrow[d, "\mathrm{id}"] & \\ \{\pm 1\} = A_{x_6} & A_{(x_6,\xi_6)} \arrow[l, "s_2 \leftarrow (s_2,s_3)"'] \arrow[r, "(s_2,s_3)\mapsto s_3"] & A_{\xi_6} = \{\pm 1\} \end{tikzcd}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \mathsf{Loc}_H(C_6) & \to & \mathsf{Loc}_H(T^*_{C_6}(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C_6} & \mapsto & \mathbb{1}_{\mathcal{O}_6} \\ & & \mathcal{L}_{\mathcal{O}_6} \\ \mathcal{F}_{C_6} & \mapsto & \mathcal{F}_{\mathcal{O}_6} \\ & & \mathcal{E}_{\mathcal{O}_6} \end{array}$$

C_7: Base point for $T^*_{C_7}(V_\lambda)_{\mathrm{sreg}}$:

$$(x_7,\xi_7) = \left(\begin{array}{c|cc|cc|c} & 1 & 0 & & & \\ \hline 0 & & & 0 & 1 & \\ 0 & & & 1 & 0 & \\ \hline & 0 & 0 & & & 0 \\ & 0 & 0 & & & 1 \\ \hline & & & 0 & 0 & \end{array}\right)$$

The equivariant fundamental group of $T^*_{C_7}(V_\lambda)_{\mathrm{sreg}}$ is $A_{(x_7,\xi_7)} = Z_{H_\lambda}((x_7,\xi_7)) = S[2]$. Thus, $T^*_{C_7}(V_\lambda)_{\mathrm{reg}}$ carries four local systems.

$\mathsf{Loc}_{H_\lambda}(T^*_{C_7}(V_\lambda)_{\mathrm{sreg}})$:	$\mathbb{1}_{\mathcal{O}_7}$	$\mathcal{L}_{\mathcal{O}_7}$	$\mathcal{F}_{\mathcal{O}_7}$	$\mathcal{E}_{\mathcal{O}_7}$
$\mathsf{Rep}(A_{(x_7,\xi_7)})$:	$++$	$--$	$-+$	$+-$

The map on equivariant fundamental groups $A_{(x_7,\xi_7)} \to A_{x_7}$ induced from the projection $T^*_{C_7}(V)_{\mathrm{sreg}} \to C_7$ is the identity, while the map on equivariant fundamental groups $A_{(x_7,\xi_7)} \to A_{\xi_7}$ induced from the projection $T^*_{C_7}(V)_{\mathrm{sreg}} \to C_7^* = C_0^t$ is trivial.

$$\begin{tikzcd} & S[2] \arrow[d, "\mathrm{id}"] & \\ A_{x_7} & A_{(x_7,\xi_7)} \arrow[l, "\mathrm{id}"'] \arrow[r] & A_{\xi_7} = 1 \end{tikzcd}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \mathsf{Loc}_H(C_7) & \to & \mathsf{Loc}_H(T^*_{C_7}(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C_7} & \mapsto & \mathbb{1}_{\mathcal{O}_7} \\ \mathcal{L}_{C_7} & \mapsto & \mathcal{L}_{\mathcal{O}_7} \\ \mathcal{F}_{C_7} & \mapsto & \mathcal{F}_{\mathcal{O}_7} \\ \mathcal{E}_{C_7} & \mapsto & \mathcal{E}_{\mathcal{O}_7} \end{array}$$

6.2.6. *Vanishing cycles of perverse sheaves.* Table 6.2.6.1 records the functor Ev on simple objects, from two perspectives.

We now give a few examples showing how to make the calculations for Table 6.2.6.1.

(a) Rows 1–5 and row 11 of Table 6.2.6.1 follow from Section 5.2.5.

(b) We show how to compute row 6. As recalled in Section 0.2.6, $\mathsf{Ev}_C\,\mathcal{IC}(\mathbb{1}_{C_4}) = 0$ unless $C \subset \overline{C}_4$, and $\mathsf{Ev}_{C_4}(\mathbb{1}_{C_4}) = \mathcal{IC}(\mathbb{1}_{\mathcal{O}_4})$ by [7, Theorem 5.3.1 (g)]. So next we determine $\mathsf{Ev}_{C_i}\,\mathbb{1}_{C_4}$ for $i = 0, 1, 2$. Recall the cover $\pi_4^{(1)} : \widetilde{C}_4^{(1)} \to \overline{C}_4$ from Section 6.2.3. As explained in Section 0.2.6, we begin by finding the singularities of the composition $(\,\cdot\,|\,\cdot\,) \circ (\pi_4^{(1)} \times \mathrm{id})$ on $\widetilde{C}_4^{(1)} \times C_i^*$. The equations that define $\widetilde{C}_4^{(1)} \times C_i^*$ as a subvariety of $V \times \mathbb{P}^1 \times V^*$ with coordinates $(w, X, [a : b], w', X')$ are

$$\begin{pmatrix} -b & a \end{pmatrix} w = 0, \quad \begin{pmatrix} a & b \end{pmatrix} X = 0,$$
$$^t w X = 0, \qquad \det(X) = 0$$

together with the equations that define C_i^* in terms of w' and X'. The conormal bundle to this variety is generated by the differentials of the functions

$$\begin{pmatrix} -b & a \end{pmatrix} w = 0, \qquad \begin{pmatrix} a & b \end{pmatrix} X = 0,$$

together with the equations that define $\overline{C_i^*}$. Thus, to find the singular locus of $(\,\cdot\,|\,\cdot\,) \circ (\pi_4^{(1)} \times \mathrm{id})$ on $\widetilde{C}_4^{(1)} \times C_i^*$. we examine the Jacobian of these functions and check to see when its rank is less than maximal. As explained in Section 0.2.6, this will determine the support of the sheaf

$$\mathsf{R}\Phi_{(\,\cdot\,|\,\cdot\,)\circ(\pi_4^{(1)}\times\mathrm{id})}(\mathbb{1}_{\widetilde{C}_4^{(1)}\times C_i^*}), \tag{39}$$

which is a sheaf on the zero locus of $(\,\cdot\,|\,\cdot\,) \circ (\pi_4^{(1)} \times \mathrm{id})$. We show below that $(\,\cdot\,|\,\cdot\,) \circ (\pi_4^{(1)} \times \mathrm{id})$ is smooth on $\widetilde{C}_4^{(1)} \times C_i^*$ for $i = 0, 1, 2$; thus $\mathsf{Ev}_{C_i}\,\mathbb{1}_{C_4} = 0$ for $i = 0, 1, 2$. We now show the remaining calculations for row 6.

$(i = 0)$ Consider the case $C_i = C_0$. The singularities of $(\,\cdot\,|\,\cdot\,)\circ(\pi_4^{(1)}\times\mathrm{id})$ on $\widetilde{C}_4^{(1)}\times C_0^*$, are found by examining the Jacobian for the functions

$$\begin{pmatrix} -b & a \end{pmatrix} w, \quad \begin{pmatrix} a & b \end{pmatrix} X, \quad w'w + \mathrm{trace}(X'X).$$

The Jacobian for those equations, in order, is below.

du	dv	dx	dy	dz	da	db	du'	dv'	dx'	dy'	dz'
$-b$	a	0	0	0	v	$-u$	0	0	0	0	0
0	0	$-a$	0	b	$-x$	z	0	0	0	0	0
0	0	0	b	a	z	y	0	0	0	0	0
u'	v'	x'	y'	$2z'$	0	0	u	v	x	y	$2z$

This system of equations form an H-bundle over $\mathbb{P}^1$, so we can specialize the $[a : b]$ coordinates to $[1 : 0]$ without loss of generality. Now we can see that if the rank of this matrix is less than 4 on $\widetilde{C}_4^{(1)} \times C_i^*$ then $u' = y' = 0$, which implies $^t w' X' w' = 0$, which contradicts $(w', X') \in C_0^*$. Since $(\,\cdot\,|\,\cdot\,) \circ (\pi_4^{(1)} \times \mathrm{id})$ is smooth on $\widetilde{C}_4^{(1)} \times C_0^*$, the vanishing cycles sheaf (39) is 0. Therefore, $\mathsf{Ev}_{C_0}\,\mathcal{IC}(\mathbb{1}_{C_4}) = 0$.

$(i = 1)$ Now consider the case $C_i = C_1$. To find the singularities of $(\,\cdot\,|\,\cdot\,) \circ (\pi_4^{(1)} \times \mathrm{id})$ on $\widetilde{C}_4^{(1)} \times C_1^*$ we simply add the equation that defines $\overline{C_1^*}$ to the list of functions in the case above. The Jacobian for the functions

$$\begin{pmatrix} -b & a \end{pmatrix} w, \quad \begin{pmatrix} a & b \end{pmatrix} X, \quad w'w + \mathrm{trace}(X'X), \qquad w' = 0,$$

is below.

du	dv	dx	dy	dz	da	db	du'	dv'	dx'	dy'	dz'
$-b$	a	0	0	0	v	$-u$	0	0	0	0	0
0	0	$-a$	0	b	$-x$	z	0	0	0	0	0
0	0	0	b	a	z	y	0	0	0	0	0
u'	v'	x'	y'	$2z'$	0	0	u	v	x	y	$2z$
0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0

Arguing as above, by setting $[a : b] = [1 : 0]$ we find $x = z = u = 0$. If the rank of this Jacobian were less than 6 then $u' = y' = 0$ so ${}^t wX'w = 0$, which would force the point to be non-regular in the conormal bundle. It follows that $(\,\cdot\,|\,\cdot\,) \circ (\pi_4^{(1)} \times \mathrm{id})$ is smooth on the regular part of $\widetilde{C}_4^{(1)} \times C_1^*$. Therefore, $\mathsf{Ev}_{C_1} \mathcal{IC}(\mathbb{1}_{C_4}) = 0$.

$(i = 2)$ The closed equation that cuts out $\overline{C_2^*}$ is $\operatorname{rank} X' = 1$. Thus, to find the singular locus of $(\,\cdot\,|\,\cdot\,) \circ (\pi_4^{(1)} \times \mathrm{id})$ on $\widetilde{C}_4^{(1)} \times C_2^*$ we consider the functions

$$\begin{pmatrix} -b & a \end{pmatrix} w, \quad \begin{pmatrix} a & b \end{pmatrix} X, \quad w'w + \operatorname{trace}(X'X), \qquad \det X',$$

and the associated Jacobian, below.

du	dv	dx	dy	dz	da	db	du'	dv'	dx'	dy'	dz'
$-b$	a	0	0	0	v	$-u$	0	0	0	0	0
0	0	$-a$	0	b	$-x$	z	0	0	0	0	0
0	0	0	b	a	z	y	0	0	0	0	0
u'	v'	x'	y'	$2z'$	0	0	u	v	x	y	$2z$
0	0	0	0	0	0	0	0	0	y'	x'	$2z'$

If the rank of this Jacobian is not maximal, then $u' = y' = 0$, which implies ${}^t w'X'w' = 0$ which contradicts $(w', X') \in C_2^*$. Thus, $(\,\cdot\,|\,\cdot\,) \circ (\pi_4^{(1)} \times \mathrm{id})$ is smooth on $\widetilde{C}_4^{(1)} \times C_2^*$. It follows that $\mathsf{Ev}_{C_2} \mathcal{IC}(\mathbb{1}_{C_4}) = 0$.

This completes the calculations needed for row 6 of Table 6.2.6.1.

(c) We show how to compute row 12. As recalled in Section 0.2.6, $\mathsf{Ev}_C \mathcal{IC}(\mathcal{F}_{C_4}) = 0$ unless $C \subset \overline{C}_4$, and $\mathsf{Ev}_{C_4} \mathcal{IC}(\mathcal{F}_{C_4}) = \mathcal{IC}(\mathcal{F}_{\mathcal{O}_4})$; see Section 6.2.5. So here we determine $\mathsf{Ev}_{C_i} \mathcal{IC}(\mathcal{F}_{C_4})$ for $i = 0, 1, 2$. Recall the cover $\pi_4^{(3)} : \widetilde{C}_4^{(3)} \to \overline{C}_4$ from Section 6.2.3. As above, we begin by finding the singularities of the composition $(\,\cdot\,|\,\cdot\,) \circ (\pi_4^{(3)} \times \mathrm{id})$ on $\widetilde{C}_4^{(3)} \times C_i^*$. The equations that define $\widetilde{C}_4^{(3)} \times C_i^*$ as a subvariety of $V \times \mathbb{A}^2 \times \mathbb{P}^1 \times V^*$ with coordinates $(w, X, A, B, [a : b], w', X')$ are

$$\begin{aligned} &\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0, && \begin{pmatrix} a & b \end{pmatrix} X = 0, \\ &\begin{pmatrix} -b & a \end{pmatrix} w = 0, && X = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \\ &\begin{pmatrix} -B & A \end{pmatrix} w = 0, \quad {}^t wX = 0, && \det(X) = 0, \end{aligned}$$

together with the equations that define C_i^* in terms of w' and X'. The conormal bundle to this variety is generated by the differentials of the functions

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \qquad \begin{pmatrix} -b & a \end{pmatrix} w = 0,$$

together with the equations that define $\overline{C_i^*}$. We find the singular locus of $(\,\cdot\,|\,\cdot\,)\circ(\pi_4^{(3)}\times\mathrm{id})$ on $\widetilde{C}_4^{(3)}\times C_i^*$ by checking the rank of the Jacobian of these functions. This will determine the support of the sheaf

$$\mathsf{R}\Phi_{(\,\cdot\,|\,\cdot\,)\circ(\pi_4^{(3)}\times\mathrm{id})}(\mathbb{1}_{\widetilde{C}_4^{(3)}\times C_i^*}). \tag{40}$$

If $(\,\cdot\,|\,\cdot\,)\circ(\pi_4^{(3)}\times\mathrm{id})$ is smooth on $\widetilde{C}_4^{(3)}\times C_i^*$ or if the restriction of this sheaf to the preimage of $(\overline{C}_4\times C_i^*)_{\mathrm{reg}}$ under $\pi_4^{(3)}\times\mathrm{id}$ is 0, then $\mathsf{Ev}_{C_i}\mathcal{F}_{C_4}=0$. However, if the restriction of (40) to the preimage of $(\overline{C}_4\times C_i^*)_{\mathrm{reg}}$ under $\pi_4^{(3)}\times\mathrm{id}$ is not 0, then to determine $\mathsf{Ev}_{C_i}\mathcal{IC}(\mathcal{F}_{C_4})$ we must calculate the pushforward of this restriction along the proper morphism $\pi_4^{(3)}\times\mathrm{id}$ (and in principle eliminate any contribution from $\mathsf{Ev}_{C_i}(\mathbb{1}_{C_4})$, however in each of the following three cases there is none). We now show the remaining calculations for row 12.

($i=2$) To find the support of (40) when $C_i=C_2$, we consider the differentials of the following functions.

$$\begin{pmatrix}-b & a\end{pmatrix}w,\quad \begin{pmatrix}a & b\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix},\quad w'w+\mathrm{Tr}(X'X),\quad \det X'.$$

This gives the following Jacobian, in which we hide x, y and z since $x=-A^2$, $z=AB$ and $y=B^2$. In this table the rows are the differentials of the above functions, in that order.

du	dv	dA	dB	da	db	du'	dv'	dx'	dy'	dz'
$-b$	a	0	0	v	$-u$	0	0	0	0	0
0	0	a	b	A	B	0	0	0	0	0
u'	v'	$2(-Ax'+Bz')$	$2(Az'+By')$	0	0	u	v	$-A^2$	B^2	$2AB$
0	0	0	0	0	0	0	0	y'	x'	$2z'$

Again we observe that this system of equations is an H-bundle over $\mathbb{P}^1$ and therefore we can set $[a:b]=[1:0]$ without loss of generality. If we do this we find $v=x=z=A=0$. Moreover, if we suppose that the rank is not maximal, then $u'=0$ by inspecting the first four columns and $y'=0$ by inspecting the fourth column. This implies ${}^tw'X'w'=0$ with contradicts $(w',X')\in C_2^*$. Thus, the singular locus of $(\,\cdot\,|\,\cdot\,)\circ(\pi_4^{(3)}\times\mathrm{id})$ on $\widetilde{C}_4^{(3)}\times C_2^*$ is empty. It follows that $\mathsf{Ev}_{C_2}\mathcal{IC}(\mathcal{F}_4)=0$.

($i=1$) To find the support of (40) when $C_i=C_1$, we consider the differentials of the following functions.

$$\begin{pmatrix}-b & a\end{pmatrix}w,\quad \begin{pmatrix}a & b\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix},\quad \mathrm{Tr}(X'X).$$

In this case we have $u'=v'=0$, so they may be omitted, and so the relevant Jacobian is:

du	dv	dA	dB	da	db	dx'	dy'	dz'
$-b$	a	0	0	v	$-u$	0	0	0
0	0	a	b	A	B	0	0	0
0	0	$2(-Ax'+Bz')$	$2(Az'+By')$	0	0	$-A^2$	B^2	$2AB$

On $\widetilde{C}_4^{(3)} \times C_1^*$ we can compute that the singular locus of $(\,\cdot\,|\,\cdot\,) \circ (\pi_4^{(3)} \times \mathrm{id})$ is cut out by

$$A = B = 0,\ \begin{pmatrix} -b & a \end{pmatrix} w = 0.$$

Note, this is already sufficient to conclude that $\mathsf{Ev}_{C_1} \mathcal{IC}(\mathcal{F}_{C_4}) \neq 0$.

Since we only need to compute the vanishing cycles (40) over the regular part of the conormal bundle, we may assume $w \neq 0$. We claim that local coordinates for the regular part of the conormal bundle are given by (X', w). Indeed, the coordinate $[a : b]$ is determined by w and all other coordinates are zero on the singular locus. It follows from this that the map from the singular locus to $T^*_C(V_\lambda)_{reg}$ is one to one. Moreover, we are free to localize away from the exceptional divisor of the blowup and thus essentially ignore $[a : b]$ while computing the vanishing cycles. Doing this, we can give new coordinates for our variety by setting

$$\begin{pmatrix} A \\ B \end{pmatrix} = cw$$

for some new coordinate c. That is, on this open we have local coordinates u, v, c, x', y', z', with no relations, and we wish to compute

$$R\Phi_{c^2(-u^2x'+2uvz'+v^2z')}(\mathbb{1}).$$

The function $-u^2x'+2uvz'+v^2z'$ is smooth and non-vanishing (on the regular part of the conormal bundle), so by setting $f = -u^2x' + 2uvz' + v^2z'$, we may consider the smooth map on our open subvariety induced from the map $\mathbb{A}^6 \to \mathbb{A}^2$ given on coordinates by $(u, v, c, x', y', z') \mapsto (c, f)$. By smooth base change $R\Phi_{c^2(-u^2x'+2uvz'+v^2z')}(\mathbb{1})$ is the pullback of $R\Phi_{c^2 f}(\mathbb{1})$. It can be shown that $R\Phi_{c^2 f}(\mathbb{1})$ is the skyscraper sheaf on $c = 0$ associated to the cover arising from taking the square root of f. Pulling this back, we have the same. This is the cover associated to the sheaf $\mathcal{F}_{\mathcal{O}_1}$ in Section 6.2.5, so $\mathsf{Ev}_{C_1} \mathcal{IC}(\mathcal{F}_{C_4}) = \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1})$.

$(i = 0)$ To find the support of (40) when $C_i = C_0$, we consider the differentials of the following functions.

$$\begin{pmatrix} -b & a \end{pmatrix} w, \quad \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \quad w'w + \mathrm{Tr}(X'X).$$

This determines the following Jacobian:

du	dv	dA	dB	da	db	du'	dv'	dx'	dy'	dz'
$-b$	a	0	0	v	$-u$	0	0	0	0	0
0	0	a	b	A	B	0	0	0	0	0
u'	v'	$2(-Ax' + Bz')$	$2(Az' + By')$	0	0	u	v	$-A^2$	B^2	$2AB$

The singular locus of $(\,\cdot\,|\,\cdot\,) \circ (\pi_4^{(3)} \times \mathrm{id})$ on $\widetilde{C}_4^{(3)} \times C_0^*$ is

$$u = v = A = B = 0,\ \begin{pmatrix} a & b \end{pmatrix} w' = 0.$$

Note, this is already sufficient to conclude that $\mathsf{Ev}_{C_1} \mathcal{IC}(\mathcal{F}_{C_4}) \neq 0$.

We may assume $w' \neq 0$, since we only need to compute (40) the vanishing cycles over the regular part of the conormal bundle. Local coordinates for the conormal bundle are now given by (w', X'). Since $[a : b]$ is determined by w', and all other coordinates are zero, it follows that the map from the

singular locus to $T^*_C(V)_{\mathrm{reg}}$ is one to one. In the following, wherever we write (a,b) you should interpret this as either $(1,b)$ or $(a,1)$ as though we were working in one of the two charts for $\mathbb{P}^1$.
We pick new local coordinates in a neighbourhood of the singular locus: these will be $[a:b], c, d, X', w'$ with the change of coordinates given by $(A,B) = c(-b,a)$ and $(u,v) = d(a,b)$. The function $ww' + \mathrm{trace}(XX')$ may now be re-written in the form

$$d\begin{pmatrix} a & b \end{pmatrix} w' + c^2 \begin{pmatrix} -b & a \end{pmatrix} X' \begin{pmatrix} -b \\ a \end{pmatrix}.$$

The functions $f = \begin{pmatrix} a & b \end{pmatrix} w'$ and $g = \begin{pmatrix} -b & a \end{pmatrix} X' \begin{pmatrix} -b \\ a \end{pmatrix}$ are smooth (on the regular part of the conormal bundle). We may thus consider the map to $\mathbb{A}^4$ induced by:

$$([a:b], c, d, X', w') \mapsto (c, d, f, g)$$

The map $ww' + \mathrm{trace}(XX')$ is simply the pullback of $df + c^2 g$.
Thus, if we can compute $R\Phi_{df+c^2g}(\mathbb{1})$ on $\mathbb{A}^4$, by smooth base change this will give us $R\Phi_{w'w+\mathrm{trace}(X'X)}(\mathbb{1})$ over the regular part of the conormal bundle.
Again, it can be shown that $R\Phi_{df+c^2g}(\mathbb{1})$ is the skyscraper sheaf over $d = f = c = 0$ associated to the cover coming from adjoining the square root of g. Pulling this back to $\widetilde{C}_4^{(3)} \times C_0^*$ and identifying the singular locus with the regular part of the conormal, we conclude that $\mathsf{Ev}_{C_0} \mathcal{IC}(\mathcal{F}_{C_4}) = \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0})$ by comparing the covers associated to the local systems in Section 6.2.5.

We close Section 6.2.8 by briefly discussing a different approach to making these calculations and then illustrate this approach by showing an alternate calculation of $\mathsf{Ev}_{C_1} \mathcal{IC}(\mathcal{F}_{C_4}) = \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1})$.

We have already demonstrated that it is reasonably straight forward to compute the support of the sheaf Ev_C by computing the singular locus in the appropriate cover. The challenging thing is computing the actual sheaf, because it tends to require adhoc changes of coordinates to understand the local structure of the singularity. However, it is still typically straight forward to compute the rank of the resulting local system. This is because we can compute this by passing to a finite étale cover which trivializes the sheaf, and by passing to an arbitrary Zariski open. In every case in this paper, when the rank is non-zero, one can immediately deduce that the rank of $\mathsf{Ev}_C \mathcal{P}$ will be 1 on the basis that the local structure is an isolated singularity direct product with an affine space.

Once one knows that the rank of $\mathsf{Ev}_C \mathcal{P}$ is 1, we need only compute the action of the fundamental group on the sheaf to identify the sheaf, and because the rank is 1, this is equivalent to computing the trace of the action of the elements of the equivariant fundamental group. We would like to illustrate now how the Lefshetz fixed point formula may be used to make these calculations. We remark that $\mathsf{Ev}_C \mathcal{P}$ is concentrated in degree 5, whereas the sheaf $\mathcal{P}$ is concentrated in degree 6.

In this section, what we need to compute is the trace of the actions of the two elements $s(-1,1)$ and $s(1,-1)$ on $W := \overline{C}_4 \times C_1^*$, noting that these determine the action of the central element $s(-1,-1)$. Because $s(-1,1)$ and $s(1,-1)$ differ by a central element, they have the same fixed point set. The fixed point set of s acting on W is

$$W^s = \mathrm{Spec}(k[x,y,v,x',y']/(xy, xv))$$

TABLE 6.2.6.1. ${}^p\mathsf{Ev} : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}})$ on simple objects, for $\lambda : W_F \to {}^LG$ given at the beginning of Section 6.

$$
\begin{array}{ccl}
\mathsf{Per}_H(V) & \xrightarrow{{}^p\mathsf{Ev}} & \mathsf{Per}_H(T^*_H(V)_{\mathrm{reg}}) \\
\mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) \\
\mathcal{IC}(\mathbb{1}_{C_1}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_1}) \\
\mathcal{IC}(\mathbb{1}_{C_2}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}) \\
\mathcal{IC}(\mathbb{1}_{C_3}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_3}) \\
\mathcal{IC}(\mathcal{L}_{C_3}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathbb{1}_{\mathcal{O}_2}) \\
\mathcal{IC}(\mathbb{1}_{C_4}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_4}) \\
\mathcal{IC}(\mathbb{1}_{C_5}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_5}) \\
\mathcal{IC}(\mathbb{1}_{C_6}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_6}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_1}) \\
\mathcal{IC}(\mathbb{1}_{C_7}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_7}) \\
\mathcal{IC}(\mathcal{L}_{C_7}) & \mapsto & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_6}) \\
\mathcal{IC}(\mathcal{F}_{C_2}) & \mapsto & \mathcal{IC}(\mathcal{F}_{\mathcal{O}_2}) \\
\mathcal{IC}(\mathcal{F}_{C_4}) & \mapsto & \mathcal{IC}(\mathcal{F}_{\mathcal{O}_4}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0}) \\
\mathcal{IC}(\mathcal{F}_{C_6}) & \mapsto & \mathcal{IC}(\mathcal{E}_{\mathcal{O}_6}) \\
\mathcal{IC}(\mathcal{F}_{C_7}) & \mapsto & \mathcal{IC}(\mathcal{F}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_5}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_3}) \\
\mathcal{IC}(\mathcal{E}_{C_7}) & \mapsto & \mathcal{IC}(\mathcal{E}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_6}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_5}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_4}) \\
 & & \oplus\, \mathcal{IC}(\mathcal{F}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})
\end{array}
$$

$\mathcal{P}$	$\mathsf{Evs}_{C_0}\mathcal{P}$	$\mathsf{Evs}_{C_1}\mathcal{P}$	$\mathsf{Evs}_{C_2}\mathcal{P}$	$\mathsf{Evs}_{C_3}\mathcal{P}$	$\mathsf{Evs}_{C_4}\mathcal{P}$	$\mathsf{Evs}_{C_5}\mathcal{P}$	$\mathsf{Evs}_{C_6}\mathcal{P}$	$\mathsf{Evs}_{C_7}\mathcal{P}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$++$	0	0	0	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_1})$	0	$++$	0	0	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_2})$	$--$	0	$--$	0	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_3})$	0	0	0	$++$	0	0	0	0
$\mathcal{IC}(\mathcal{L}_{C_3})$	0	0	$++$	$--$	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_4})$	0	0	0	0	$+$	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_5})$	0	0	0	0	0	$+$	0	0
$\mathcal{IC}(\mathbb{1}_{C_6})$	0	$--$	0	0	0	0	$--$	0
$\mathcal{IC}(\mathbb{1}_{C_7})$	0	0	0	0	0	0	0	$++$
$\mathcal{IC}(\mathcal{L}_{C_7})$	0	0	0	0	0	0	$++$	$--$
$\mathcal{IC}(\mathcal{F}_{C_2})$	0	0	$+-$	0	0	0	0	0
$\mathcal{IC}(\mathcal{F}_{C_4})$	$-+$	$-+$	0	0	$-$	0	0	0
$\mathcal{IC}(\mathcal{F}_{C_6})$	0	0	0	0	0	0	$+-$	0
$\mathcal{IC}(\mathcal{F}_{C_7})$	0	0	0	$-+$	0	$-$	0	$-+$
$\mathcal{IC}(\mathcal{E}_{C_7})$	$+-$	$-+$	$-+$	$-+$	$-$	$-$	$-+$	$+-$

and the restriction of $f := (\,\cdot\mid\cdot\,)$ to W^s is $f = xx' + yy'$. Now, the restriction of $\mathcal{IC}(\mathcal{F}_{C_4}) \boxtimes \mathbb{1}_{C_1^*}$ to W^s is sheaf associated to pullback of the original cover to this fixed point set, so it appears in the proper pushforward formed by adjoining $x = A^2, y = B^2$:

$$\widetilde{W^s} = \mathrm{Spec}(k[A, B, v, x', y']/(AB, B^2 v)).$$

In these coordinates, $f^s = A^2x' + B^2y'$. The map $\widetilde{W^s} \to W^s$ is an equivariant cover which admits a non-trivial action of $s(-1, 1)$ and $s(1, -1)$, where the former acts as $A \mapsto -A$

and the later acts as $B \mapsto -B$. By observing that the map $\widetilde{W^s} \to W^s$ restricts to a bijection on the singular locus of f^s, we may conclude that the trace of the action on stalks of the cover will agree with that on the base.

Though it is possible by working with local coordinates to compute the vanishing cycles here as we did above, we will instead use the Lefshetz theorem again. There are now two cases:

$s(-1,1)$: Recall that map $\widetilde{W^s} \to W^s$ is an equivariant cover which admits a non-trivial action of $s(-1,1)$ given by $A \mapsto -A$, in the coordinates above. The fix of this action on $\widetilde{W^s}$ and the restriction of f^s to that fix, is

$$\mathrm{Spec}(k[B,v,x',y']/(B^2v)) \qquad f^s = B^2y'.$$

We note that this cover is of relative dimension 1. Noting further that we wish to evaluate the action over regular conormal vectors this imposes the condition $u \neq 0$ which implies $B = 0$, and leads us to consider:

$$\mathrm{Spec}(k[v,x',y']) \qquad f^s = 0$$

which allows us to conclude that the vanishing cycles are the constant sheaf in degree 6 (because the singular locus is codimension 0) and so the trace is 1, accounting for the shift of hte original sheaf leads to a trace of -1.

$s(1,-1)$: The action of $s(-1,1)$ on the cover $\widetilde{W^s} \to W^s$ is given by $B \mapsto -B$ so restriction to the fix of that action forces $B = 0$ Thus, the fix of the action of $s(-1,1)$ on $\widetilde{W^s}$ and the restriction of f^s to that fix, is

$$\mathrm{Spec}(k[A,v,x',y']) \qquad f^s = A^2x'.$$

We note that this is relative dimension 0 and that we know, as above, that the vanishing cycles are associated to the cover coming from taking $\sqrt{x'}$. However, $s(1,-1)$ acts trivially on this cover and so again we conclude the trace is 1, the sheaf here is concentrated in degree 5 (because the singular locus is codimension 1), so the trace on the original sheaf is still 1.

Checking the association between characters of the fundamental group and sheaves on the regular conormal bundle allows us to conclude $\mathsf{Ev}_{C_1}\mathcal{IC}(\mathcal{F}_{C_4}) = \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1})$, as above.

6.2.7. *Normalization of Ev and the twisting local system.* Using Table 6.2.6.1 we find our second case when the equivariant local system $\mathcal{T}$ is non-trivial:

$$\mathcal{T} = \mathbb{1}^\sharp_{\mathcal{O}_0} \oplus \mathbb{1}^\sharp_{\mathcal{O}_1} \oplus \mathcal{L}^\sharp_{\mathcal{O}_2} \oplus \mathbb{1}^\sharp_{\mathcal{O}_3} \oplus \mathbb{1}^\sharp_{\mathcal{O}_4} \oplus \mathbb{1}^\sharp_{\mathcal{O}_5} \oplus \mathcal{L}^\sharp_{\mathcal{O}_6} \oplus \mathbb{1}^\sharp_{\mathcal{O}_7}.$$

We use $\mathcal{T}$ in Table 6.2.7.1 to calculate ${}^p\mathsf{NEv} : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}})$ in two forms; compare with Table 6.2.6.1

6.2.8. *Vanishing cycles and the Fourier transform.* Compare Table 6.2.8.1 with the Fourier transform from Section 6.2.4 to confirm (21) in this example.

TABLE 6.2.7.1. ${}^p\mathsf{NEv} : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}})$ on simple objects, for $\lambda : W_F \to {}^LG$ given at the beginning of Section 6.

$\mathsf{Per}_H(V)$	$\overset{{}^p\mathsf{NEv}}{\longrightarrow}$	$\mathsf{Per}_H(T^*_H(V)_{\mathrm{reg}})$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathbb{1}_{C_1})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_1})$
$\mathcal{IC}(\mathbb{1}_{C_2})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathbb{1}_{C_3})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_3})$
$\mathcal{IC}(\mathcal{L}_{C_3})$	$\mapsto$	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2})$
$\mathcal{IC}(\mathbb{1}_{C_4})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_4})$
$\mathcal{IC}(\mathbb{1}_{C_5})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_5})$
$\mathcal{IC}(\mathbb{1}_{C_6})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_6}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_1})$
$\mathcal{IC}(\mathbb{1}_{C_7})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_7})$
$\mathcal{IC}(\mathcal{L}_{C_7})$	$\mapsto$	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_6})$
$\mathcal{IC}(\mathcal{F}_{C_2})$	$\mapsto$	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}_2})$
$\mathcal{IC}(\mathcal{F}_{C_4})$	$\mapsto$	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_4}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathcal{F}_{C_6})$	$\mapsto$	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_6})$
$\mathcal{IC}(\mathcal{F}_{C_7})$	$\mapsto$	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_5}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_3})$
$\mathcal{IC}(\mathcal{E}_{C_7})$	$\mapsto$	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_6}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_5}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_4})$
		$\oplus\, \mathcal{IC}(\mathcal{F}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})$

$\mathcal{P}$	$\mathsf{NEvs}_{C_0}\mathcal{P}$	$\mathsf{NEvs}_{C_1}\mathcal{P}$	$\mathsf{NEvs}_{C_2}\mathcal{P}$	$\mathsf{NEvs}_{C_3}\mathcal{P}$	$\mathsf{NEvs}_{C_4}\mathcal{P}$	$\mathsf{NEvs}_{C_5}\mathcal{P}$	$\mathsf{NEvs}_{C_6}\mathcal{P}$	$\mathsf{NEvs}_{C_7}\mathcal{P}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$++$	0	0	0	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_1})$	0	$++$	0	0	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_2})$	$--$	0	$++$	0	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_3})$	0	0	0	$++$	0	0	0	0
$\mathcal{IC}(\mathcal{L}_{C_3})$	0	0	$--$	$--$	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_4})$	0	0	0	0	$+$	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_5})$	0	0	0	0	0	$+$	0	0
$\mathcal{IC}(\mathbb{1}_{C_6})$	0	$--$	0	0	0	0	$++$	0
$\mathcal{IC}(\mathbb{1}_{C_7})$	0	0	0	0	0	0	0	$++$
$\mathcal{IC}(\mathcal{L}_{C_7})$	0	0	0	0	0	0	$--$	$--$
$\mathcal{IC}(\mathcal{F}_{C_2})$	0	0	$-+$	0	0	0	0	0
$\mathcal{IC}(\mathcal{F}_{C_4})$	$-+$	$-+$	0	0	$-$	0	0	0
$\mathcal{IC}(\mathcal{F}_{C_6})$	0	0	0	0	0	0	$-+$	0
$\mathcal{IC}(\mathcal{F}_{C_7})$	0	0	0	$-+$	0	$-$	0	$-+$
$\mathcal{IC}(\mathcal{E}_{C_7})$	$+-$	$-+$	$+-$	$-+$	$-$	$-$	$+-$	$+-$

TABLE 6.2.8.1. Fourier transform and vanishing cycles

$\mathsf{Per}_{H_\lambda}(V_\lambda)$	$\xrightarrow{{}^p\mathsf{NEv}}$	$\mathsf{Per}_H(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}})$	$\xrightarrow{a_*}$	$\mathsf{Per}_H(T^*_{H_\lambda}(V^*_\lambda)_{\mathrm{reg}})$	$\xleftarrow{{}^p\mathsf{Ev}^*}$	$\mathsf{Per}_{H_\lambda}(V^*_\lambda)$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}^*_0})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathbb{1}_{C^*_0})$
$\mathcal{IC}(\mathbb{1}_{C_1})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_1})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}^*_1})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathbb{1}_{C^*_1})$
$\mathcal{IC}(\mathbb{1}_{C_2})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}^*_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}^*_0})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathcal{L}_{C^*_0})$
$\mathcal{IC}(\mathbb{1}_{C_3})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_3})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}^*_3})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathbb{1}_{C^*_3})$
$\mathcal{IC}(\mathcal{L}_{C_3})$	$\mapsto$	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2})$	$\mapsto$	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}^*_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}^*_2})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathbb{1}_{C^*_2})$
$\mathcal{IC}(\mathbb{1}_{C_4})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_4})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}^*_4})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathbb{1}_{C^*_4})$
$\mathcal{IC}(\mathbb{1}_{C_5})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_5})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}^*_5})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathbb{1}_{C^*_5})$
$\mathcal{IC}(\mathbb{1}_{C_6})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_6}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_1})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}^*_6}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}^*_1})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathcal{L}_{C^*_1})$
$\mathcal{IC}(\mathbb{1}_{C_7})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_7})$	$\mapsto$	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}^*_7})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathbb{1}_{C^*_7})$
$\mathcal{IC}(\mathcal{L}_{C_7})$	$\mapsto$	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_6})$	$\mapsto$	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}^*_7}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}^*_6})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathbb{1}_{C^*_6})$
$\mathcal{IC}(\mathcal{F}_{C_2})$	$\mapsto$	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}_2})$	$\mapsto$	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}^*_2})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathcal{F}_{C^*_2})$
$\mathcal{IC}(\mathcal{F}_{C_4})$	$\mapsto$	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_4}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1})$ $\oplus\, \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0})$	$\mapsto$	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}^*_4}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}^*_1})$ $\oplus\, \mathcal{IC}(\mathcal{F}_{\mathcal{O}^*_0})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathcal{F}_{C^*_0})$
$\mathcal{IC}(\mathcal{F}_{C_6})$	$\mapsto$	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_6})$	$\mapsto$	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}^*_6})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathcal{F}_{C^*_6})$
$\mathcal{IC}(\mathcal{F}_{C_7})$	$\mapsto$	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_5})$ $\oplus\, \mathcal{IC}(\mathcal{F}_{\mathcal{O}_3})$	$\mapsto$	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}^*_7}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}^*_5})$ $\oplus\, \mathcal{IC}(\mathcal{F}_{\mathcal{O}^*_3})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathcal{F}_{C^*_5})$
$\mathcal{IC}(\mathcal{E}_{C_7})$	$\mapsto$	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_6})$ $\oplus\, \mathcal{IC}(\mathcal{E}_{\mathcal{O}_5}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_4})$ $\oplus\, \mathcal{IC}(\mathcal{F}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_2})$ $\oplus\, \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})$	$\mapsto$	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}^*_7}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}^*_6})$ $\oplus\, \mathcal{IC}(\mathcal{E}_{\mathcal{O}^*_5}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}^*_4})$ $\oplus\, \mathcal{IC}(\mathcal{F}_{\mathcal{O}^*_3}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}^*_2})$ $\oplus\, \mathcal{IC}(\mathcal{F}_{\mathcal{O}^*_1}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}^*_0})$	$\leftarrow\!\shortmid$	$\mathcal{IC}(\mathcal{E}_{C^*_0})$

6.2.9. *Arthur sheaves.* Arthur perverse sheaves in $\mathsf{Per}_{H_\lambda}(V_\lambda)$, decomposed into pure packet sheaves and coronal perverse sheaves, are displayed below.

Arthur sheaves	pure packet sheaves	coronal sheaves
$\mathcal{A}_{C_0}$	$\mathcal{IC}(\mathbb{1}_{C_0}) \oplus$	$\mathcal{IC}(\mathbb{1}_{C_2}) \oplus \mathcal{IC}(\mathcal{F}_{C_4}) \oplus \mathcal{IC}(\mathcal{E}_{C_7})$
$\mathcal{A}_{C_1}$	$\mathcal{IC}(\mathbb{1}_{C_1}) \oplus$	$\mathcal{IC}(\mathbb{1}_{C_6}) \oplus \mathcal{IC}(\mathcal{F}_{C_4}) \oplus \mathcal{IC}(\mathcal{E}_{C_7})$
$\mathcal{A}_{C_2}$	$\mathcal{IC}(\mathbb{1}_{C_2}) \oplus \mathcal{IC}(\mathcal{F}_{C_2}) \oplus$	$\mathcal{IC}(\mathcal{L}_{C_3}) \oplus \mathcal{IC}(\mathcal{E}_{C_7})$
$\mathcal{A}_{C_3}$	$\mathcal{IC}(\mathbb{1}_{C_3}) \oplus \mathcal{IC}(\mathcal{L}_{C_3}) \oplus$	$\mathcal{IC}(\mathcal{F}_{C_7}) \oplus \mathcal{IC}(\mathcal{E}_{C_7})$
$\mathcal{A}_{C_4}$	$\mathcal{IC}(\mathbb{1}_{C_4}) \oplus \mathcal{IC}(\mathcal{F}_{C_4}) \oplus$	$\mathcal{IC}(\mathcal{E}_{C_7})$
$\mathcal{A}_{C_5}$	$\mathcal{IC}(\mathbb{1}_{C_5}) \oplus$	$\mathcal{IC}(\mathcal{F}_{C_7}) \oplus \mathcal{IC}(\mathcal{E}_{C_7})$
$\mathcal{A}_{C_6}$	$\mathcal{IC}(\mathbb{1}_{C_6}) \oplus \mathcal{IC}(\mathcal{F}_{C_6}) \oplus$	$\mathcal{IC}(\mathcal{L}_{C_7}) \oplus \mathcal{IC}(\mathcal{E}_{C_7})$
$\mathcal{A}_{C_7}$	$\mathcal{IC}(\mathbb{1}_{C_7}) \oplus \mathcal{IC}(\mathcal{L}_{C_7}) \oplus \mathcal{IC}(\mathcal{F}_{C_7}) \oplus \mathcal{IC}(\mathcal{E}_{C_7})$	

6.3. **Adams-Barbasch-Vogan packets.**

6.3.1. *Admissible representations versus perverse sheaves.* Using Vogan's bijection between $\mathsf{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{/\mathrm{iso}}$ and $\Pi_{\mathrm{pure},\lambda}(G/F)$ as discussed in Section 0.3.1, we now match the 8 Langlands parameters from Section 6.1.1 with the 8 strata from Section 6.2.1 and the

15 admissible representations from Section 6.1.2 with the 15 perverse sheaves from Section 6.2.3.

$\mathsf{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}}$	$\Pi_{\text{pure},\lambda}(G/F)$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$[\pi(\phi_0), 0]$
$\mathcal{IC}(\mathbb{1}_{C_1})$	$[\pi(\phi_1), 0]$
$\mathcal{IC}(\mathbb{1}_{C_2})$	$[\pi(\phi_2, +), 0]$
$\mathcal{IC}(\mathbb{1}_{C_3})$	$[\pi(\phi_3, +), 0]$
$\mathcal{IC}(\mathcal{L}_{C_3})$	$[\pi(\phi_3, -), 0]$
$\mathcal{IC}(\mathbb{1}_{C_4})$	$[\pi(\phi_4, +), 0)]$
$\mathcal{IC}(\mathbb{1}_{C_5})$	$[\pi(\phi_5), 0]$
$\mathcal{IC}(\mathbb{1}_{C_6})$	$[\pi(\phi_6, +), 0]$
$\mathcal{IC}(\mathbb{1}_{C_7})$	$[\pi(\phi_7, ++), 0]$
$\mathcal{IC}(\mathcal{L}_{C_7})$	$[\pi(\phi_7, --), 0]$
$\mathcal{IC}(\mathcal{F}_2)$	$[\pi(\phi_2, -), 1]$
$\mathcal{IC}(\mathcal{F}_4)$	$[\pi(\phi_4, -), 1]$
$\mathcal{IC}(\mathcal{F}_6)$	$[\pi(\phi_6, -), 1]$
$\mathcal{IC}(\mathcal{F}_7)$	$[\pi(\phi_7, -+), 1]$
$\mathcal{IC}(\mathcal{E}_7)$	$[\pi(\phi_7, +-), 1]$

6.3.2. *ABV-packets.* Using the bijection from Section 6.3.1 and the calculation of the functor Ev from Section 6.2.6, we now easily find the ABV-packets $\Pi^{\text{ABV}}_{\text{pure},\phi}(G/F)$ for Langlands parameters ϕ with infinitesimal parameter $\lambda : W_F \to {}^LG$, using Section 0.3.2. In each case we find the pure L-packet $\Pi_{\text{pure},\phi}(G/F)$ and $\Pi^{\text{ABV}}_{\text{pure},\phi}(G/F)$ and the remaining coronal representations:

ABV-packet	pure L-packet	coronal representations
$\Pi^{\text{ABV}}_{\text{pure},\phi_0}(G/F)$	$[\pi(\phi_0), 0]$	$[\pi(\phi_2, +), 0], [\pi(\phi_4, -), 1], [\pi(\phi_7, +-), 1]$
$\Pi^{\text{ABV}}_{\text{pure},\phi_1}(G/F)$	$[\pi(\phi_1), 0]$	$[\pi(\phi_4, -), 1], [\pi(\phi_6, +), 0], [\pi(\phi_7, +-), 1]$
$\Pi^{\text{ABV}}_{\text{pure},\phi_2}(G/F)$	$[\pi(\phi_2, +), 0], [\pi(\phi_2, -), 1]$	$[\pi(\phi_3, -), 0], [\pi(\phi_7, +-), 1]$
$\Pi^{\text{ABV}}_{\text{pure},\phi_3}(G/F)$	$[\pi(\phi_3, +), 0], [\pi(\phi_3, -), 0]$	$[\pi(\phi_7, -+), 1], [\pi(\phi_7, +-), 1]$
$\Pi^{\text{ABV}}_{\text{pure},\phi_4}(G/F)$	$[\pi(\phi_4, +), 0],\ [\pi(\phi_4, -), 1]$	$[\pi(\phi_7, +-), 1]$
$\Pi^{\text{ABV}}_{\text{pure},\phi_5}(G/F)$	$[\pi(\phi_5), 0]$	$[\pi(\phi_7, -+), 1],\ [\pi(\phi_7, +-), 1]$
$\Pi^{\text{ABV}}_{\text{pure},\phi_6}(G/F)$	$[\pi(\phi_6, +), 0],\ [\pi(\phi_6, -), 1]$	$[\pi(\phi_7, --), 0],\ [\pi(\phi_7, +-), 1]$
$\Pi^{\text{ABV}}_{\text{pure},\phi_7}(G/F)$	$[\pi(\phi_7, ++), 0],\ [\pi(\phi_7, --), 0],\ [\pi(\phi_7, -+), 1],\ [\pi(\phi_7, +-), 1]$	

We record the stable distributions $\eta^{\mathsf{NEv}}_{\psi}$ arising from ABV-packets through our calculations. We will examine the invariant distributions $\eta^{\mathsf{NEv}}_{\psi,s}$, later.

ABV-packets	pure L-packet representations	coronal representations
$\eta^{\mathsf{NEv}}_{\psi_0}$	$+\pi(\phi_0)$	$+\pi(\phi_2,+)+\pi(\phi_4,-)+\pi(\phi_7,+-)$
$\eta^{\mathsf{NEv}}_{\phi_1}$	$+\pi(\phi_1)$	$+\pi(\phi_4,-)+\pi(\phi_6,+)+\pi(\phi_7,+-)$
$\eta^{\mathsf{NEv}}_{\psi_2}$	$+\pi(\phi_2,+)-\pi(\phi_2,-)$	$-\pi(\phi_3,-)+\pi(\phi_7,+-)$
$\eta^{\mathsf{NEv}}_{\phi_3}$	$+\pi(\phi_3,+)+\pi(\phi_3,-)$	$-\pi(\phi_7,-+)-\pi(\phi_7,+-)$
$\eta^{\mathsf{NEv}}_{\psi_4}$	$+\pi(\phi_4,+)-\pi(\phi_4,-)$	$-\pi(\phi_7,+-)$
$\eta^{\mathsf{NEv}}_{\psi_5}$	$+\pi(\phi_5)$	$+\pi(\phi_7,-+)+\pi(\phi_7,+-)$
$\eta^{\mathsf{NEv}}_{\psi_6}$	$+\pi(\phi_6,+)-\pi(\phi_6,-)$	$-\pi(\phi_7,--)+\pi(\phi_7,+-)$
$\eta^{\mathsf{NEv}}_{\psi_7}$	$+\pi(\phi_7,++)+\pi(\phi_7,--)-\pi(\phi_7,-+)-\pi(\phi_7,+-)$	

6.3.3. *Kazhdan-Lusztig conjecture.* Using the bijection of Section 6.1.4, we compare the multiplicity matrix from Section 6.1.3

$$m_{\mathrm{rep}} = \left(\begin{array}{cccccccccc|cccc|c}
1&1&1&1&1&2&2&1&1&1&0&0&0&0&0\\
0&1&0&0&0&1&1&1&1&1&0&0&0&0&0\\
0&0&1&1&0&1&1&1&1&0&0&0&0&0&0\\
0&0&0&1&0&0&1&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&1&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&1&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right)$$

with the normalised geometric multiplicity matrix from Section 6.2.3.

$$m'_{\mathrm{geo}} = \left(\begin{array}{cccccccccc|cccc|c}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
2&1&1&0&0&1&0&0&0&0&0&0&0&0&0\\
2&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&0&0&1&0&1&0&0&0&0&0&0&0\\
1&1&1&1&0&1&1&1&1&0&0&0&0&0&0\\
1&1&0&0&1&0&1&0&0&1&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&1&1&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right)$$

Since ${}^t m_{\mathrm{rep}} = m'_{\mathrm{geo}}$, this confirms the Kazhdan-Lusztig conjecture in this example.

6.3.4. *Aubert duality and Fourier transform.* To verify (30), use Vogan's bijection from Section 6.3.1 to compare Aubert duality from Section 6.1.5 with the Fourier transform from Section 6.2.4.

To verify (31), compare the twisting characters χ_ψ of A_ψ from Section 6.1.5 with the restriction $\mathcal{T}_\psi$ to $T^*_{C_\psi}(V)_{\mathrm{reg}}$ of the $\mathcal{T}$ from Section 6.2.8. In both cases the character is trivial except on A_{ψ_2} and A_{ψ_6}, where it is the character $(--)$, using notation from Section 6.1.4. Using this notation, here is another perspective on $\mathcal{T}$, where for each $C \subseteq V$ we display the corresponding character of A_C^{mic}.

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
$\mathcal{T}_C$	$++$	$++$	$--$	$++$	$+$	$+$	$--$	$++$

6.3.5. *ABV-packets that are not Arthur packets.* We conclude Section 6.3 by drawing attention to the two ABV-packets $\Pi^{\mathrm{ABV}}_{\mathrm{pure},\phi_1}(G/F)$ and $\Pi^{\mathrm{ABV}}_{\mathrm{pure},\phi_3}(G/F)$ that are not Arthur packets, as ϕ_1 and ϕ_3 are not of Arthur type. While the following two admissible homomorphisms $L_F \times \mathrm{SL}(2,\mathbb{C}) \to {}^L G$ are not Arthur parameters because they are not bounded on W_F,

$$\begin{aligned}
\psi_1(w,x,y) &:= \nu_2(y) \oplus (\nu_2^2(d_w) \otimes \nu_2(x)),\\
\psi_3(w,x,y) &:= \nu_2(x) \oplus (\nu_2^2(d_w) \otimes \nu_2(y)),
\end{aligned}$$

they do behave like Arthur parameters in other regards, as we now explain. First $\phi_{\psi_1} = \phi_1$ and $\phi_{\psi_3} = \phi_3$. We note too that ψ_3 is the Aubert dual of ψ_1. Let us define

$$\Pi_{\mathrm{pure},\psi_1}(G/F) := \Pi^{\mathrm{ABV}}_{\mathrm{pure},\phi_1}(G/F) \qquad \text{and} \qquad \Pi_{\mathrm{pure},\psi_3}(G/F) := \Pi^{\mathrm{ABV}}_{\mathrm{pure},\phi_3}(G/F).$$

Then $\Pi_{\mathrm{pure},\psi_1}(G/F)$ and $\Pi_{\mathrm{pure},\psi_3}(G/F)$ define the following pseudo-Arthur packets for G_1 and G_0:

$$\begin{aligned}
\Pi_{\psi_1}(G_0(F)) &:= \{\pi(\phi_1),\ \pi(\phi_6,+)\},\\
\Pi_{\psi_3}(G_0(F)) &:= \{\pi(\phi_3,+),\ \pi(\phi_3,-)\},
\end{aligned}$$

and

$$\begin{aligned}
\Pi_{\psi_1}(G_1(F)) &:= \{\pi(\phi_4,-),\ \pi(\phi_7,+-)\},\\
\Pi_{\psi_3}(G_1(F)) &:= \{\pi(\phi_7,-+), \pi(\phi_7,+-)\}.
\end{aligned}$$

Aubert duality defines a bijection between $\Pi_{\psi_3}(G_0(F))$ and $\Pi_{\psi_1}(G_0(F))$ and between $\Pi_{\psi_3}(G_1(F))$ and $\Pi_{\psi_1}(G_1(F))$. Moreover, it follows from the Kazhdan-Lusztig conjecture, which we have already established for this example in Section 6.3.3, that the associated distributions

$$\begin{aligned}\Theta^{G_0}_{\psi_1} &:= \operatorname{trace}\pi(\phi_1) + \operatorname{trace}\pi(\phi_6,+)\\ \Theta^{G_0}_{\psi_3} &:= \operatorname{trace}\pi(\phi_3,+) + \operatorname{trace}\pi(\phi_3,-)\end{aligned}$$

and

$$\begin{aligned}\Theta^{G_1}_{\psi_1} &:= -\left(-\operatorname{trace}\pi(\phi_4,-) - \operatorname{trace}\pi(\phi_7,+-)\right)\\ \Theta^{G_1}_{\psi_3} &:= -\left(+\operatorname{trace}\pi(\phi_7,-+) + \operatorname{trace}\pi(\phi_7,+-)\right)\end{aligned}$$

are stable. Moreover, using the characters of microlocal fundamental groups arising from our calculation of the functor Ev_{C_1} and Ev_{C_3} we may define $\Theta^{G_0}_{\psi_1,s}$, $\Theta^{G_1}_{\psi_1,s}$, $\Theta^{G_0}_{\psi_1,s}$ and $\Theta^{G_0}_{\psi_1,s}$. It follows from Section 6.1.6 that these distributions coincide with the endoscopic transfer of stable distributions from an elliptic endoscopic group G'; those stable distributions on $G'(F)$ also arise from ABV-packets that are not Arthur packets. In these regards, the pseudo-Arthur packets $\Pi_{\psi_1}(G_0(F))$, $\Pi_{\psi_1}(G_1(F))$, $\Pi_{\psi_3}(G_0(F))$, and $\Pi_{\psi_3}(G_1(F))$ behave like Arthur packets.

6.4. **Endoscopy and equivariant restriction of perverse sheaves.** In this section we will calculate both sides of (34) for $G = \mathrm{SO}(7)$ and the elliptic endoscopic $G' = \mathrm{SO}(5)\times \mathrm{SO}(3)$, which already appeared in Section 6.1.6. This will illustrate how the Langlands-Shelstad lift of $\Theta_{\psi'}$ on $G'(F)$ to $\Theta_{\psi,s}$ on $G(F)$ is related to equivariant restriction of perverse sheaves from V to the Vogan variety V' for G'; see Section 6.1.6 for ψ'.

The endoscopic datum for G' includes $s \in H$ given by

$$s := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that

$$Z_{\widehat{G}}(s) = \left\{ \left(\begin{array}{c|c|c} A & 0 & B\\ \hline 0 & E & 0\\ \hline C & 0 & D \end{array}\right) \mid \begin{pmatrix} A & B\\ C & D\end{pmatrix} \in \mathrm{Sp}(4), E \in \mathrm{Sp}(2)\right\} \cong \mathrm{Sp}(4)\times\mathrm{Sp}(4),$$

so $\widehat{G}' = Z_{\widehat{G}}(s)$.

6.4.1. *Endoscopic Vogan variety.* The infinitesimal parameter $\lambda : W_F \to {}^LG$ factors through $\epsilon : {}^LG' \hookrightarrow {}^LG$ to define $\lambda' : W_F \to {}^LG'$ by

$$\lambda'(w) = \left(\begin{pmatrix} |w|^{3/2} & 0 & 0 & 0\\ 0 & |w|^{1/2} & 0 & 0\\ 0 & 0 & |w|^{-1/2} & 0\\ 0 & 0 & 0 & |w|^{-3/2}\end{pmatrix}, \begin{pmatrix} |w|^{1/2} & 0\\ 0 & |w|^{-1/2}\end{pmatrix}\right).$$

To simplify notation below, let us set $G^{(1)} := \mathrm{SO}(3)$ and $G^{(2)} := \mathrm{SO}(5)$ and define $\lambda^{(1)} : W_F \to {}^LG^{(1)}$ and $\lambda^{(2)} : W_F \to {}^LG^{(2)}$ accordingly. Also set

$$H^{(1)} := Z_{\widehat{G}^{(1)}}(\lambda^{(1)}) \quad \text{and} \quad H^{(2)} := Z_{\widehat{G}^{(2)}}(\lambda^{(2)})$$

and $V^{(1)} := V_{\lambda^{(1)}}$ and $V^{(2)} := V_{\lambda^{(2)}}$. Then,

$$H' = H^{(1)} \times H^{(2)} \quad \text{and} \quad V' = V^{(2)} \times V^{(2)},$$

with the action of $H^{(1)}$ on $V^{(1)}$ given in Section 2 and the action of $H^{(2)}$ on $V^{(2)}$ given in Section 4. It follows that, with reference to Sections 2 and 4, V' is stratified into eight H'-orbits:

$$\begin{array}{cccc} C_{ux} \times C_y & C_x \times C_y & C_u \times C_y & C_0 \times C_y \\ C_{ux} \times C_0 & C_x \times C_0 & C_u \times C_0 & C_0 \times C_0. \end{array}$$

For all H'-orbits $C' \subset V'$, the microlocal fundamental group $A^{\mathrm{mic}}_{C'}$ is canonically isomorphic to the centre $Z(\widehat{G'}) = Z(\widehat{G}^{(2)}) \times Z(\widehat{G}^{(1)})$, because we have chosen G' so that the unramified infinitesimal parameter λ' is regular semisimple at Fr. Consequently, the image of $Z(\widehat{G'})$ under $\epsilon : \widehat{G'} \hookrightarrow \widehat{G}$ is the group $S[2]$ introduced in Section 6.1.4.

6.4.2. *Restriction.* We now describe the restriction functor $\mathsf{D}_H(V) \to \mathsf{D}_{H'}(V')$ on simple perverse sheaves, after passing to Grothendieck groups.

$$\begin{array}{rcl}
\mathrm{res} : \mathsf{Per}_H(V) & \longrightarrow & \mathsf{KPer}_{H'}(V') \\
\mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathbb{1}_{C_0})[0] \\
\mathcal{IC}(\mathbb{1}_{C_1}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathbb{1}_{C_0})[1] \\
\mathcal{IC}(\mathbb{1}_{C_2}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_x} \boxtimes \mathbb{1}_{C_0})[1] \oplus \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathbb{1}_{C_y})[1] \oplus \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathbb{1}_{C_0})[1] \\
\mathcal{IC}(\mathbb{1}_{C_3}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_x} \boxtimes \mathbb{1}_{C_y})[1] \\
\mathcal{IC}(\mathcal{L}_{C_3}) & \mapsto & \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathcal{E}_{C_y})[1] \oplus \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathbb{1}_{C_0})[1] \\
\mathcal{IC}(\mathbb{1}_{C_4}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_x} \boxtimes \mathbb{1}_{C_0})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathbb{1}_{C_y})[1] \\
\mathcal{IC}(\mathbb{1}_{C_5}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathbb{1}_{C_y})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_x} \boxtimes \mathbb{1}_{C_y})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathbb{1}_{C_y})[2] \\
 & & \oplus\, \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathcal{E}_{C_y})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathbb{1}_{C_0})[2] \\
\mathcal{IC}(\mathbb{1}_{C_6}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_{ux}} \boxtimes \mathbb{1}_{C_0})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathbb{1}_{C_y})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathbb{1}_{C_0})[2] \\
\mathcal{IC}(\mathbb{1}_{C_7}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_{ux}} \boxtimes \mathbb{1}_{C_y})[2] \\
\mathcal{IC}(\mathcal{L}_{C_7}) & \mapsto & \mathcal{IC}(\mathcal{L}_{C_{ux}} \boxtimes \mathcal{E}_{C_y})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathbb{1}_{C_0})[2] \\
\mathcal{IC}(\mathcal{F}_{C_2}) & \mapsto & \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathbb{1}_{C_0})[1] \oplus \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathcal{E}_{C_y})[1] \\
\mathcal{IC}(\mathcal{F}_{C_4}) & \mapsto & \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathcal{E}_{C_y})[1] \oplus \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathbb{1}_{C_0})[2] \\
\mathcal{IC}(\mathcal{F}_{C_6}) & \mapsto & \mathcal{IC}(\mathcal{L}_{C_{ux}} \boxtimes \mathbb{1}_{C_0})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathcal{E}_{C_y})[2] \\
\mathcal{IC}(\mathcal{F}_{C_7}) & \mapsto & \mathcal{IC}(\mathcal{L}_{C_{ux}} \boxtimes \mathbb{1}_{C_y})[2] \oplus \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathbb{1}_{C_y})[2] \oplus \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathbb{1}_{C_0})[4] \\
 & & \oplus\, \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathcal{E}_{C_y})[4] \\
\mathcal{IC}(\mathcal{E}_{C_7}) & \mapsto & \mathcal{IC}(\mathcal{L}_{C_{ux}} \boxtimes \mathcal{E}_{C_y})[2] \oplus \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathcal{E}_{C_y})[2]
\end{array}$$

6.4.3. *Restriction and vanishing cycles.* Although the inclusion $V' \hookrightarrow V$ induces a map of conormal bundles $\epsilon : T^*_{H'}(V') \hookrightarrow T^*_H(V)$, this does not restrict to a map of regular conormal bundles. Instead, we have

$$\begin{array}{rcl}
T^*_{C_0}(V)_{\mathrm{reg}} \cap T^*_{H'}(V')_{\mathrm{reg}} & = & T^*_{C_0 \times C_0}(V')_{\mathrm{reg}} \\
T^*_{C_1}(V)_{\mathrm{reg}} \cap T^*_{H'}(V')_{\mathrm{reg}} & = & T^*_{C_u \times C_0}(V')_{\mathrm{reg}} \\
T^*_{C_2}(V)_{\mathrm{reg}} \cap T^*_{H'}(V')_{\mathrm{reg}} & = & T^*_{C_0 \times C_y}(V')_{\mathrm{reg}} \\
T^*_{C_3}(V)_{\mathrm{reg}} \cap T^*_{H'}(V')_{\mathrm{reg}} & = & T^*_{C_x \times C_y}(V')_{\mathrm{reg}} \\
T^*_{C_4}(V)_{\mathrm{reg}} \cap T^*_{H'}(V')_{\mathrm{reg}} & = & \emptyset \\
T^*_{C_5}(V)_{\mathrm{reg}} \cap T^*_{H'}(V')_{\mathrm{reg}} & = & \emptyset \\
T^*_{C_6}(V)_{\mathrm{reg}} \cap T^*_{H'}(V')_{\mathrm{reg}} & = & T^*_{C_{ux} \times C_0}(V')_{\mathrm{reg}} \\
T^*_{C_7}(V)_{\mathrm{reg}} \cap T^*_{H'}(V')_{\mathrm{reg}} & = & T^*_{C_{ux} \times C_y}(V')_{\mathrm{reg}}
\end{array}$$

Thus, the hypothesis for (33) is met only for $(x', \xi') \in T^*_{H'}(V')_{\mathrm{reg}}$ from the list of regular conormal bundles appearing on the right-hand side of these equations.

We now prove a few more interesting cases of (33).

($\mathcal{IC}(\mathcal{E}_{C_7})$). Take $\mathcal{P} = \mathcal{IC}(\mathcal{E}_{C_7})$. From Section 6.4.2 we see that, in the Grothendieck group of $\mathsf{Per}_{H'}(T^*_{H'}(V')_{\mathrm{reg}})$,

$$\begin{aligned}
&\mathsf{Ev}'\left(\mathcal{IC}(\mathcal{E}_{C_7})|_{V'}\right)\\
\equiv\ & \mathsf{Ev}'\left(\mathcal{IC}(\mathcal{L}_{C_{ux}} \boxtimes \mathcal{E}_{C_y}) \oplus \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathcal{E}_{C_y})\right)\\
=\ & \left(\mathsf{Ev}^{(2)}\,\mathcal{IC}(\mathcal{L}_{C_{ux}}) \boxtimes \mathsf{Ev}^{(1)}\,\mathcal{IC}(\mathcal{E}_{C_y})\right) \oplus \left(\mathsf{Ev}^{(2)}\,\mathcal{IC}(\mathcal{L}_{C_x}) \boxtimes \mathsf{Ev}^{(1)}\,\mathcal{IC}(\mathcal{E}_{C_y})\right)\\
=\ & \left((\mathcal{IC}(\mathcal{L}_{\mathcal{O}_{ux}}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_u})) \boxtimes (\mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}))\right)\\
&\oplus\ \left((\mathcal{IC}(\mathcal{L}_{\mathcal{O}_x}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})) \boxtimes (\mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}))\right)\\
=\ & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_{ux}} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_{ux}} \boxtimes \mathcal{E}_{\mathcal{O}_0}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_u} \boxtimes \mathcal{E}_{\mathcal{O}_y})\\
&\oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_u} \boxtimes \mathcal{E}_{\mathcal{O}_0}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_x} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_x} \boxtimes \mathcal{E}_{\mathcal{O}_0})\\
&\oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0} \boxtimes \mathcal{E}_{\mathcal{O}_0}).
\end{aligned}$$

On the other hand, recall from Section 6.2.7 that

$$\begin{aligned}
\mathsf{Ev}\,\mathcal{IC}(\mathcal{E}_{C_7}) \quad = \quad & \mathcal{IC}(\mathcal{E}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_6}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_5}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_4})\\
& + \mathcal{IC}(\mathcal{F}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}).
\end{aligned}$$

We can now easily calculate both sides of (33) on all six components of $T^*_H(V)_{\mathrm{reg}} \cap T^*_{H'}(V')_{\mathrm{reg}}$.

($C_0 \times C_0$). If $(x', \xi') \in T^*_{C_0 \times C_0}(V')_{\mathrm{reg}}$ then $(x, \xi) \in T^*_{C_0}(V)_{\mathrm{reg}}$. In this case the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim(C_0 \times C_0)}\,\mathrm{trace}_{a'_s}\left(\mathsf{Ev}'\,\mathcal{IC}(\mathcal{E}_{C_7})|_{V'}\right)_{(x',\xi')}\\
=\ & (-1)^0\,\mathrm{trace}_{(+1,-1)}\,\mathcal{IC}(\mathcal{L}_{\mathcal{O}_0} \boxtimes \mathcal{E}_{\mathcal{O}_0})\\
=\ & (--)(+1,-1)\\
=\ & -1,
\end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C_0}\,\mathrm{trace}_{a_s}\left(\mathsf{Ev}\,\mathcal{IC}(\mathcal{E}_{C_7})\right)_{(x,\xi)}\\
=\ & (-1)^{\dim C_0}\,\mathrm{trace}_{a_s}\,\mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})\\
=\ & \mathrm{trace}_{(+1,-1)}\,\mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})\\
=\ & (+-)(+1,-1)\\
=\ & -1.
\end{aligned}$$

This confirms (33) on $T^*_{C_0 \times C_0}(V')_{\mathrm{reg}}$.

($C_u \times C_0$). If $(x', \xi') \in T^*_{C_u \times C_0}(V')_{\mathrm{reg}}$ then $(x, \xi) \in T^*_{C_1}(V)_{\mathrm{reg}}$. In this case the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim(C_u \times C_0)}\,\mathrm{trace}_{a'_s}\left(\mathsf{Ev}'\,\mathcal{IC}(\mathcal{E}_{C_7})|_{V'}\right)_{(x',\xi')}\\
=\ & (-1)^1\,\mathrm{trace}_{(+1,-1)}\,\mathcal{IC}(\mathcal{L}_{\mathcal{O}_u} \boxtimes \mathcal{E}_{\mathcal{O}_0})\\
=\ & -(--)(+1,-1)\\
=\ & +1,
\end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C_1}\,\mathrm{trace}_{a_s}\left(\mathsf{Ev}\,\mathcal{IC}(\mathcal{E}_{C_7})\right)_{(x,\xi)}\\
=\ & (-1)^{\dim C_1}\,\mathrm{trace}_{a_s}\,\mathcal{IC}(\mathcal{F}_{\mathcal{O}_1})\\
=\ & (-1)^2\,\mathrm{trace}_{(+1,-1)}\,\mathcal{IC}(\mathcal{F}_{\mathcal{O}_1})\\
=\ & (-+)(+1,-1)\\
=\ & +1.
\end{aligned}$$

This confirms (33) on $T^*_{C_u\times C_0}(V')_{\mathrm{reg}}$.

$(C_0\times C_y)$. If $(x',\xi')\in T^*_{C_0\times C_y}(V')_{\mathrm{reg}}$ then $(x,\xi)\in T^*_{C_2}(V)_{\mathrm{reg}}$. In this case the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim(C_0\times C_y)}\operatorname{trace}_{a'_s}\left(\mathsf{Ev}'\,\mathcal{IC}(\mathcal{E}_{C_7})|_{V'}\right)_{(x',\xi')}\\
=\;&(-1)^1\operatorname{trace}_{(+1,-1)}\mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}\boxtimes\mathcal{E}_{\mathcal{O}_y})\\
=\;&-(--)(+1,-1)\\
=\;&+1,
\end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C_2}\operatorname{trace}_{a_s}\left(\mathsf{Ev}\,\mathcal{IC}(\mathcal{E}_{C_7})\right)_{(x,\xi)}\\
=\;&(-1)^{\dim C_2}\operatorname{trace}_{a_s}\mathcal{IC}(\mathcal{F}_{\mathcal{O}_2})\\
=\;&(-1)^2\operatorname{trace}_{(+1,-1)}\mathcal{IC}(\mathcal{F}_{\mathcal{O}_2})\\
=\;&(-+)(+1,-1)\\
=\;&+1.
\end{aligned}$$

This confirms (33) on $T^*_{C_0\times C_y}(V')_{\mathrm{reg}}$.

$(C_x\times C_y)$. If $(x',\xi')\in T^*_{C_x\times C_y}(V')_{\mathrm{reg}}$ then $(x,\xi)\in T^*_{C_3}(V)_{\mathrm{reg}}$. In this case the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim(C_x\times C_y)}\operatorname{trace}_{a'_s}\left(\mathsf{Ev}'\,\mathcal{IC}(\mathcal{E}_{C_7})|_{V'}\right)_{(x',\xi')}\\
=\;&(-1)^2\operatorname{trace}_{(+1,-1)}\mathcal{IC}(\mathcal{L}_{\mathcal{O}_x}\boxtimes\mathcal{E}_{\mathcal{O}_y})\\
=\;&(--)(+1,-1)\\
=\;&-1,
\end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C_3}\operatorname{trace}_{a_s}\left(\mathsf{Ev}\,\mathcal{IC}(\mathcal{E}_{C_7})\right)_{(x,\xi)}\\
=\;&(-1)^{\dim C_3}\operatorname{trace}_{a_s}\mathcal{IC}(\mathcal{F}_{\mathcal{O}_3})\\
=\;&(-1)^3\operatorname{trace}_{(+1,-1)}\mathcal{IC}(\mathcal{F}_{\mathcal{O}_3})\\
=\;&-(-+)(+1,-1)\\
=\;&-1.
\end{aligned}$$

This confirms (33) on $T^*_{C_x\times C_y}(V')_{\mathrm{reg}}$.

$(C_{ux}\times C_0)$. If $(x',\xi')\in T^*_{C_{ux}\times C_0}(V')_{\mathrm{reg}}$ then $(x,\xi)\in T^*_{C_6}(V)_{\mathrm{reg}}$. In this case the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim(C_{ux}\times C_0)}\operatorname{trace}_{a'_s}\left(\mathsf{Ev}'\,\mathcal{IC}(\mathcal{E}_{C_7})|_{V'}\right)_{(x',\xi')}\\
=\;&(-1)^2\operatorname{trace}_{(+1,-1)}\mathcal{IC}(\mathcal{L}_{\mathcal{O}_{ux}}\boxtimes\mathcal{E}_{\mathcal{O}_0})\\
=\;&(--)(+1,-1)\\
=\;&-1,
\end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C_6}\operatorname{trace}_{a_s}\left(\mathsf{Ev}\,\mathcal{IC}(\mathcal{E}_{C_7})\right)_{(x,\xi)}\\
=\;&(-1)^{\dim C_6}\operatorname{trace}_{a_s}\mathcal{IC}(\mathcal{E}_{\mathcal{O}_6})\\
=\;&(-1)^4\operatorname{trace}_{(+1,-1)}\mathcal{IC}(\mathcal{E}_{\mathcal{O}_6})\\
=\;&(+-)(+1,-1)\\
=\;&-1.
\end{aligned}$$

This confirms (33) on $T^*_{C_{ux}\times C_0}(V')_{\mathrm{reg}}$.

$(C_{ux} \times C_y)$. If $(x', \xi') \in T^*_{C_{ux} \times C_y}(V')_{\mathrm{reg}}$ then $(x, \xi) \in T^*_{C_7}(V)_{\mathrm{reg}}$. In this case the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim(C_{ux} \times C_y)} \operatorname{trace}_{a'_s} \left(\mathsf{Ev}' \, \mathcal{IC}(\mathcal{E}_{C_7})|_{V'}\right)_{(x', \xi')} \\
= \quad & (-1)^3 \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}_{ux}} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \\
= \quad & -(--)(+1,-1) \\
= \quad & +1,
\end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C_7} \operatorname{trace}_{a_s} \left(\mathsf{Ev} \, \mathcal{IC}(\mathcal{E}_{C_7})\right)_{(x,\xi)} \\
= \quad & (-1)^{\dim C_7} \operatorname{trace}_{a_s} \mathcal{IC}(\mathcal{E}_{\mathcal{O}_7}) \\
= \quad & (-1)^5 \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{E}_{\mathcal{O}_7}) \\
= \quad & -(+-)(+1,-1) \\
= \quad & +1.
\end{aligned}$$

This confirms (33) on $T^*_{C_{ux} \times C_y}(V')_{\mathrm{reg}}$.

This confirms (34) for $\mathcal{P} = \mathcal{IC}(\mathcal{E}_{C_7})$.

$(\mathcal{IC}(\mathcal{F}_{C_4})$. Here is another interesting example of (34). Take $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_4})$. Then from Section 6.4.2 we see that, in the Grothendieck group of $\mathsf{Per}_{H'}(T^*_{H'}(V')_{\mathrm{reg}})$,

$$\begin{aligned}
&\mathsf{Ev}' \left(\mathcal{IC}(\mathcal{F}_{C_4})|_{V'}\right) \\
\equiv \quad & \mathsf{Ev}' \left(\mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathcal{E}_{C_y})[1] \oplus \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathbb{1}_{C_0})\right) \\
= \quad & \left(\mathsf{Ev}^{(2)} \, \mathcal{IC}(\mathbb{1}_{C_u}) \boxtimes \mathsf{Ev}^{(1)} \, \mathcal{IC}(\mathcal{E}_{C_y})\right) \oplus \left(\mathsf{Ev}^{(2)} \, \mathcal{IC}(\mathcal{L}_{C_x}) \boxtimes \mathsf{Ev}^{(1)} \, \mathcal{IC}(\mathbb{1}_{C_0})\right) \\
= \quad & \left(\mathcal{IC}(\mathbb{1}_{\mathcal{O}_u}) \boxtimes (\mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}))\right) \oplus \left((\mathcal{IC}(\mathcal{L}_{\mathcal{O}_x}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})) \boxtimes \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})\right) \\
= \quad & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_u} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathbb{1}_{\mathcal{O}_u} \boxtimes \mathcal{E}_{\mathcal{O}_0}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_x} \boxtimes \mathcal{L}_{\mathcal{O}_0}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0} \boxtimes \mathbb{1}_{\mathcal{O}_0}).
\end{aligned}$$

On the other hand, recall from Section 6.2.7 that

$$\mathsf{Ev} \, \mathcal{IC}(\mathcal{F}_{C_4}) \quad = \quad \mathcal{IC}(\mathcal{F}_{\mathcal{O}_4}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0}).$$

We can now easily calculate both sides of (33) on all six components of $T^*_H(V)_{\mathrm{reg}} \cap T^*_{H'}(V')_{\mathrm{reg}}$.

$(C_0 \times C_0)$. If $(x', \xi') \in T^*_{C_0 \times C_0}(V')_{\mathrm{reg}}$ then the left-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim(C_0 \times C_0)} \operatorname{trace}_{a'_s} \left(\mathsf{Ev}' \, \mathcal{IC}(\mathcal{F}_{C_4})|_{V'}\right)_{(x', \xi')} \\
= \quad & (-1)^0 \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0} \boxtimes \mathbb{1}_{\mathcal{O}_0}) \\
= \quad & (-+)(+1,-1) \\
= \quad & +1,
\end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}
&(-1)^{\dim C_0} \operatorname{trace}_{a_s} \left(\mathsf{Ev} \, \mathcal{IC}(\mathcal{F}_{C_4})\right)_{(x,\xi)} \\
= \quad & (-1)^{\dim C_0} \operatorname{trace}_{a_s} \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0}) \\
= \quad & \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0}) \\
= \quad & (-+)(+1,-1) \\
= \quad & +1.
\end{aligned}$$

This confirms (33) on $T^*_{C_0 \times C_0}(V')_{\mathrm{reg}}$ for $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_4})$.

$(C_u \times C_0)$. If $(x', \xi') \in T^*_{C_u \times C_0}(V')_{\mathrm{reg}}$ then $(x, \xi) \in T^*_{C_1}(V)_{\mathrm{reg}}$. In this case the left-hand side of (33) is

$$\begin{aligned} & (-1)^{\dim(C_u \times C_0)} \operatorname{trace}_{a'_s} \left(\mathsf{Ev}' \, \mathcal{IC}(\mathcal{F}_{C_4})|_{V'}\right)_{(x',\xi')} \\ = \quad & (-1)^1 \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathbb{1}_{\mathcal{O}_u} \boxtimes \mathcal{E}_{\mathcal{O}_0}) \\ = \quad & -(+-)(+1,-1) \\ = \quad & +1, \end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned} & (-1)^{\dim C_1} \operatorname{trace}_{a_s} \left(\mathsf{Ev} \, \mathcal{IC}(\mathcal{F}_{C_4})\right)_{(x,\xi)} \\ = \quad & (-1)^{\dim C_1} \operatorname{trace}_{a_s} \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \\ = \quad & (-1)^2 \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \\ = \quad & (-+)(+1,-1) \\ = \quad & +1. \end{aligned}$$

This confirms (33) on $T^*_{C_u \times C_0}(V')_{\mathrm{reg}}$ for $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_4})$.

$(C_0 \times C_y)$. If $(x', \xi') \in T^*_{C_0 \times C_y}(V')_{\mathrm{reg}}$ then $(x, \xi) \in T^*_{C_2}(V)_{\mathrm{reg}}$. In this case the left-hand side of (33) is

$$\begin{aligned} & (-1)^{\dim(C_0 \times C_y)} \operatorname{trace}_{a'_s} \left(\mathsf{Ev}' \, \mathcal{IC}(\mathcal{F}_{C_4})|_{V'}\right)_{(x',\xi')} \\ = \quad & (-1)^1 \operatorname{trace}_{(+1,-1)} 0 \\ = \quad & 0, \end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned} & (-1)^{\dim C_2} \operatorname{trace}_{a_s} \left(\mathsf{Ev} \, \mathcal{IC}(\mathcal{F}_{C_4})\right)_{(x,\xi)} \\ = \quad & (-1)^{\dim C_2} \operatorname{trace}_{a_s} 0 \\ = \quad & 0. \end{aligned}$$

This confirms (33) on $T^*_{C_0 \times C_y}(V')_{\mathrm{reg}}$ for $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_4})$.

$(C_x \times C_y)$. If $(x', \xi') \in T^*_{C_x \times C_y}(V')_{\mathrm{reg}}$ then $(x, \xi) \in T^*_{C_3}(V)_{\mathrm{reg}}$. In this case the left-hand side of (33) is

$$\begin{aligned} & (-1)^{\dim(C_x \times C_y)} \operatorname{trace}_{a'_s} \left(\mathsf{Ev}' \, \mathcal{IC}(\mathcal{F}_{C_4})|_{V'}\right)_{(x',\xi')} \\ = \quad & (-1)^2 \operatorname{trace}_{(+1,-1)} 0 \\ = \quad & 0, \end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned} & (-1)^{\dim C_3} \operatorname{trace}_{a_s} \left(\mathsf{Ev} \, \mathcal{IC}(\mathcal{F}_{C_4})\right)_{(x,\xi)} \\ = \quad & (-1)^{\dim C_3} \operatorname{trace}_{a_s} 0 \\ = \quad & 0. \end{aligned}$$

This confirms (33) on $T^*_{C_x \times C_y}(V')_{\mathrm{reg}}$ for $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_4})$.

$(C_{ux} \times C_0)$. If $(x', \xi') \in T^*_{C_{ux} \times C_0}(V')_{\mathrm{reg}}$ then $(x, \xi) \in T^*_{C_6}(V)_{\mathrm{reg}}$. In this case the left-hand side of (33) is

$$\begin{aligned} & (-1)^{\dim(C_{ux} \times C_0)} \operatorname{trace}_{a'_s} \left(\mathsf{Ev}' \, \mathcal{IC}(\mathcal{F}_{C_4})|_{V'}\right)_{(x',\xi')} \\ = \quad & (-1)^2 \operatorname{trace}_{(+1,-1)} 0 \\ = \quad & 0, \end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}&(-1)^{\dim C_6}\operatorname{trace}_{a_s}\left(\mathsf{Ev}\,\mathcal{IC}(\mathcal{F}_{C_4})\right)_{(x,\xi)}\\ =\;&(-1)^{\dim C_6}\operatorname{trace}_{a_s}0\\ =\;&0.\end{aligned}$$

This confirms (33) on $T^*_{C_{ux}\times C_0}(V')_{\mathrm{reg}}$ for $\mathcal{P}=\mathcal{IC}(\mathcal{F}_{C_4})$.

$(C_{ux}\times C_y)$. If $(x',\xi')\in T^*_{C_{ux}\times C_y}(V')_{\mathrm{reg}}$ then $(x,\xi)\in T^*_{C_7}(V)_{\mathrm{reg}}$. In this case the left-hand side of (33) is

$$\begin{aligned}&(-1)^{\dim(C_{ux}\times C_y)}\operatorname{trace}_{a'_s}\left(\mathsf{Ev}'\,\mathcal{IC}(\mathcal{E}_{C_7})|_{V'}\right)_{(x',\xi')}\\ =\;&(-1)^3\operatorname{trace}_{(+1,-1)}0\\ =\;&0,\end{aligned}$$

while the right-hand side of (33) is

$$\begin{aligned}&(-1)^{\dim C_7}\operatorname{trace}_{a_s}\left(\mathsf{Ev}\,\mathcal{IC}(\mathcal{F}_{C_4})\right)_{(x,\xi)}\\ =\;&(-1)^{\dim C_7}\operatorname{trace}_{a_s}0\\ =\;&0.\end{aligned}$$

This confirms (33) on $T^*_{C_{ux}\times C_y}(V')_{\mathrm{reg}}$ for $\mathcal{P}=\mathcal{IC}(\mathcal{F}_{C_4})$.

This confirms (34) for $\mathcal{P}=\mathcal{IC}(\mathcal{E}_{C_7})$.

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