# Bernstein centre for enhanced Langlands parameters

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# Bernstein decomposition

Let $G$ be a connected reductive group over a $p$-adic field $F$. The set of (equivalences classes of) irreducibles representations of $G$ is decomposed as

$$\mathrm{Irr}(G) = \bigsqcup_{\mathfrak{s}\in\mathcal{B}(G)} \mathrm{Irr}(G)_{\mathfrak{s}},$$

where $\mathfrak{s} = [M, \sigma]$ with $M$ a Levi subgroup of $G$ and $\sigma \in \mathrm{Irr}(M)$ cuspidal. There is a map $\mathbf{Sc} : \mathrm{Irr}(G) \longrightarrow \Omega(G)$ which associate to an irreducible representation its cuspidal support.

**Question**

How to define the Bernstein decomposition for Langlands parameters ? What is the notion of cuspidal Langlands parameter ? of cuspidal support ?

# Generalized Springer correspondence

Let $H$ be a complex connected reductive group
For all $x \in H$, we denote $A_H(x) = Z_H(x)/Z_H(x)^\circ$.

$$\mathcal{N}_H^+ = \left\{ (\mathcal{C}_u^H, \eta) \middle| \; u \in H \text{ unipotent}, \eta \in \mathrm{Irr}(A_H(u)) \right\}$$

We denote by $\mathcal{S}_H$ the set of ($H$-conjugacy classes of) triples $(L, \mathcal{C}_v^L, \varepsilon)$ with

- $L$ a Levi subgroup of $H$ ;
- $\mathcal{C}_v^L$ an unipotent $L$-orbit ;
- $\varepsilon \in \mathrm{Irr}(A_L(v))$ cuspidal.

- For all $H$, the triple $(T, \{1\}, 1) \in \mathcal{S}_H$, and $N_H(T)/T$ is the Weyl group of $H$ ;

| $H$ | condition | unipotent orbi | $A_H(u)$ | $\varepsilon$ |
|---|---|---|---|---|
| $\mathrm{GL}_n$ | $n = 1$ | $\mathcal{O}_{(1)}$ | $\{1\}$ | 1 |
| $\mathrm{Sp}_{2n}$ | $2n = d(d+1)$ | $\mathcal{O}_{(2d,2d-2,\ldots,4,2)}$ | $(\mathbb{Z}/2\mathbb{Z})^d$ | $\varepsilon(z_{2i}) = (-1)^i$ |
| $\mathrm{SO}_n$ | $n = d^2$ | $\mathcal{O}_{(2d-1,2d-3,\ldots,3,1)}$ | $(\mathbb{Z}/2\mathbb{Z})^{d-1}$ | $\varepsilon(z_{2i-1}z_{2i+1}) = -1$ |

Let $u \in G$ be a unipotent element and $\varepsilon \in \mathrm{Irr}(A_H(u))$.
Let $P = LU$ be a parabolic subgroup of $H$ and $v \in L$ be a unipotent element.
We define

$$Y_{u,v} = \left\{ gZ_L(v)^\circ U \mid g \in H, g^{-1}ug \in vU \right\}$$

and

$$d_{u,v} = \frac{1}{2}(\dim Z_H(u) - \dim Z_L(v)).$$

Then $\dim Y_{u,v} \leqslant d_{u,v}$ and $Z_H(u)$ acts on $Y_{u,v}$ by left translation. We denote by $S_{u,v}$ the permutation representation on the irreducibles components of $Y_{u,v}$ of dimension $d_{u,v}$.
If $P = B = TU$, then

$$Y_{u,1} = \left\{ gB \in H/B \mid g \in H, g^{-1}ug \in U \right\} = \left\{ B' \in \mathcal{B} \mid u \in B' \right\} = \mathcal{B}_u.$$

### Definition

We say that $\varepsilon$ is cuspidal, if and only if, for all proper parabolic subgroup $P = LU$, for all unipotent $v \in L$, we have $\mathrm{Hom}_{A_G(u)}(\varepsilon, S_{u,v}) = 0$.

# Generalized Springer correspondence

$$\mathcal{N}_H^+ = \left\{ (\mathcal{C}_u^H, \eta) \,\middle|\, u \in H \text{ unipotent}, \eta \in \mathrm{Irr}(A_H(u)) \right\}$$

$\mathcal{S}_H$ the set of ($H$-conjugacy classes of) triples $(L, \mathcal{C}_v^L, \varepsilon)$ with

- $L$ a Levi subgroup of $H$ ;
- $\mathcal{C}_v^L$ an unipotent $L$-orbit ;
- $\varepsilon \in \mathrm{Irr}(A_L(v))$ cuspidal.

We denote by $W_L^H = N_H(L)/L$.

### Theorem (Lusztig,1984)

$$\mathcal{N}_H^+ \simeq \bigsqcup_{(L,\mathcal{C}_v^L,\varepsilon)\in\mathcal{S}_H} \mathrm{Irr}(W_L^H)$$

$$(\mathcal{C}_u^H, \eta) \longleftrightarrow (L, \mathcal{C}_v^L, \varepsilon; \rho)$$

# Generalized Springer correspondence in a disconnected case

We suppose now that $H$ is a reductive not necessarily connected
$H$ acts by conjugation on $\mathcal{N}_{H^\circ}^{+}$ and $\mathcal{S}_{H^\circ}$.

**Proposition (M.)**

The generalized Springer correspondence for $H^\circ$ is $H$-équivariante, i.e.

$$h \cdot (\mathcal{C}_u^{H^\circ}, \eta) \longleftrightarrow h \cdot (L^\circ, \mathcal{C}_v^{L^\circ}, \varepsilon; \rho).$$

**Définition**

We call quasi-Levi subgroup of $H$ a subgroup of the form $L = Z_H(A)$, where $A$ is a torus contained in $H^\circ$.

The neutral component of a quasi-Levi subgroup of $H$ is a Levi subgroup of $H^\circ$.
$W_L^H = N_H(A)/Z_H(A)$ admits $W_{L^\circ}^{H^\circ} = N_{H^\circ}(L^\circ)/L^\circ$ as normal subgroup.

# Generalized Springer correspondence in a disconnected case

Let $u \in H^\circ$ be unipotent et $\varepsilon \in \mathrm{Irr}(A_H(u))$. We say that $\varepsilon$ is cuspidal if all irreducibles subrepresentations of $A_{H^\circ}(u)$ which appear in the restrcition to $A_{H^\circ}(u)$ are cuspidals.

We denote by

$$\mathcal{N}_H^+ = \left\{(\mathcal{C}_u^H, \eta), \;\; u \in H^\circ \text{ unipotent}, \eta \in \mathrm{Irr}(A_H(u))\right\}$$

$\mathcal{S}_H$ the set of ($H$-conjugacy classes of) triples $(L, \mathcal{C}_v^L, \varepsilon)$ avec

- $L$ quasi-Levi subgroup of $H$ ;
- $\mathcal{C}_v^L$ a unipotent $L$-orbit ;
- $\varepsilon \in \mathrm{Irr}(A_L(v))$ cuspidal.

## Theorem (M.)

For $H = O_n$,

$$\mathcal{N}_H^+ \simeq \bigsqcup_{(L, \mathcal{C}_v^L, \varepsilon) \in \mathcal{S}_H} \mathrm{Irr}(W_L^H)$$

$$(\mathcal{C}_u^H, \eta) \longleftrightarrow (L, \mathcal{C}_v^L, \varepsilon, \rho)$$

# Langlands correspondence

Let be $F$ a $p$-adic field and $G$ a split reductive connected group over $F$. We denote by $\widehat{G}$ the Langlands dual group of $G$, $W_F$ the Weil group of $F$ and $W_F' = W_F \times \mathrm{SL}_2(\mathbb{C})$ the Weil-Deligne group.

## Définition

A Langlands parameter of $G$ is a continous morphism

$$\phi : W_F' \longrightarrow \widehat{G},$$

such that

- $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic ;
- $\phi(W_F)$ consist of semisimple elements.

We denote by $\Phi(G)$ the set of $\widehat{G}$-conjugation classes of Langlands parameters of $G$.

# Langlands correspondence

We denote by $\mathrm{Irr}(G)$ the set of (smooth) irreducible representations of $G$.

## Conjecture

There exists a finite to one map

$$\mathrm{rec}_G : \mathrm{Irr}(G) \longrightarrow \Phi(G).$$

Hence,

$$\mathrm{Irr}(G) = \bigsqcup_{\phi \in \Phi(G)} \Pi_\phi(G).$$

There exists a bijection

$$\Pi_\phi(G) \simeq \mathrm{Irr}(\mathcal{S}_\phi^G),$$

avec $\mathcal{S}_\phi^G = Z_{\widehat{G}}(\phi)/Z_{\widehat{G}}(\phi)^\circ Z_{\widehat{G}}$.
+ other proprieties.

# Langlands correspondence

The Langlands correspondence is proved for $\mathrm{GL}_n$ by Harris et Taylor ; Henniart et Scholze, for $\mathrm{SO}_n$ et $\mathrm{Sp}_{2n}$ by Arthur. We denote by

$$\Phi(G)^+ = \left\{ (\phi, \eta) \middle| \phi \in \Phi(G),\, \eta \in \mathrm{Irr}(\mathcal{S}_\phi^G) \right\}.$$

Then

$$\mathrm{rec}_G^+ : \mathrm{Irr}(G) \simeq \Phi(G)^+.$$

Properties of the Langlands correspondence : for all $\phi \in \Phi(G)$, the following are equivalent

- one element in $\Pi_\phi(G)$ is in the discrete serie ;
- all elements in $\Pi_\phi(G)$ are in the discrete serie ;
- $\phi \in \Phi(G)_2$ (is discrete).

Supercuspidal ?

# Jordan bloc of discrete series

Let $G$ be one of the split groups $\mathrm{Sp}_{2n}(F)$ or $\mathrm{SO}_n(F)$.
For all unitary irreducible supercuspidal representation $\pi$ de $\mathrm{GL}_{d_\pi}(F)$ and for all integer $a \geqslant 1$, the induced representation

$$\pi|\ |^{\frac{a-1}{2}} \times \pi|\ |^{\frac{a-3}{2}} \times \ldots \times \pi|\ |^{\frac{1-a}{2}},$$

admit a unique irreducible subrepresentation of $\mathrm{GL}_{ad_\pi}(F)$ : $\mathrm{St}(\pi, a)$.
Let $\tau$ be an irreducible discrete serie of $G$. We denote by

$$\mathrm{Jord}(\tau) = \{(\pi, a)\}$$

with $\pi$ an unitary irreducible supercuspidal representation of a $\mathrm{GL}_{d_\pi}(F)$ and $a \geqslant 1$ such that there exists an integer $a'$ which verify :

$$\begin{cases} a \equiv a' \mod 2 \\ \mathrm{St}(\pi, a) \rtimes \tau \quad \text{irréductible} \\ \mathrm{St}(\pi, a') \rtimes \tau \quad \text{réductible} \end{cases}$$

# Jordan bloc of discrete series

Let $\varphi \in \Phi(G)$ a discrete parameter. The decomposition of $\mathrm{Std} \circ \phi$, where $\mathrm{Std} : \widehat{G} \hookrightarrow \mathrm{GL}_{N_{\widehat{G}}}(\mathbb{C})$ is :

$$\mathrm{Std} \circ \phi = \bigoplus_{\pi \in I_\varphi} \bigoplus_{a \in J_\pi} \pi \boxtimes S_a.$$

We call Jordan bloc of $\varphi$ and we denote by $\mathrm{Jord}(\varphi) = \{(\pi, a) | \pi \in I_\varphi, a \in J_\pi\}$.

# Jordan bloc of discrete series

*Jord*$(\varphi)$ without hole (or jump)
$\Longleftrightarrow ((\pi, a) \in \mathrm{Jord}(\varphi)$ et $a \geqslant 3 \Longrightarrow (\pi, a-2) \in \mathrm{Jord}(\varphi))$.
$A_{\widehat{G}}(\varphi)$ is generated by

$$\begin{cases} z_{\pi,a} & \text{pour } (\pi, a) \in \mathrm{Jord}(\varphi) \text{ et } a \text{ pair} \\ z_{\pi,a} z_{\pi,a'} & \text{pour } (\pi, a), (\pi, a') \in \mathrm{Jord}(\varphi) \text{ without parity condition on } a, a' \end{cases}$$

$(\pi, a), (\pi, a') \in \mathrm{Jord}(\varphi)$, with $a' < a$, consecutive $\Leftrightarrow$ for all $b \in [\![a'+1, a-1]\!]$, $(\pi, b) \notin \mathrm{Jord}(\varphi)$.
$a_{\pi,\min}$ the smallest integer $a \geqslant 1$ such that $(\pi, a) \in \mathrm{Jord}(\varphi)$.

## Définition

A character $\varepsilon$ of $A_{\widehat{G}}(\varphi)$ is alternate if for all $(\pi, a), (\pi, a') \in \mathrm{Jord}(\varphi)$ consecutive, $\varepsilon(z_{\pi,a} z_{\pi,a'}) = -1$ and if for all $(\pi, a_{\pi,\min}) \in \mathrm{Jord}(\varphi)$ with $a_{\pi,\min}$ evenn, $\varepsilon(z_{\pi,a_{\pi,\min}}) = -1$.

# Jordan bloc of discrete series

**Theorem (Mœglin)**

The Langlands classification of discrete series of $G$ by Arthur induce a bijection between the set of irreducible supercuspidal representation of $G$ and the set of pairs $(\varphi, \varepsilon)$ such that $\mathrm{Jord}(\varphi)$ is without holes and $\varepsilon$ is alternate ; the bijection $\tau \mapsto (\varphi, \varepsilon)$ is defined by $\mathrm{Jord}(\varphi) = \mathrm{Jord}(\tau)$ et $\varepsilon = \varepsilon_\tau$.

# Stable Bernstein centre, after Haines

Let $G$ be a split connected reductive $p$-adic group.
$(\widehat{M}, \lambda)$ with $\widehat{M}$ a Levi subgroup of $\widehat{G}$ et $\lambda : W_F \longrightarrow \widehat{M}$ discrete.
Unramified cocharacters $\mathfrak{X}(\widehat{M}) = \{\chi : W_F/I_F \longrightarrow Z^{\circ}_{\widehat{M}}\} \simeq Z^{\circ}_{\widehat{M}}$.

## Definition

1. the cuspidal $L$-data $(\widehat{M}_1, \lambda_1)$ and $(\widehat{M}_2, \lambda_2)$ are associate if there exists $g \in \widehat{G}$ such that ${}^g\widehat{M}_1 = \widehat{M}_2$ and $\lambda_2 = {}^g\lambda_1$ ;
2. the cuspidal $L$-data $(\widehat{M}_1, \lambda_1)$ and $(\widehat{M}_2, \lambda_2)$ are inertially equivalent if there exists $g \in \widehat{G}$ and $\chi \in \mathfrak{X}(\widehat{M}_2)$ such that ${}^g\widehat{M}_1 = \widehat{M}_2$ and $\lambda_2 = {}^g\lambda_1\chi$.

We denote by $\Omega(G)_{\mathrm{st}}$ (resp. $\mathcal{B}(G)_{\mathrm{st}}$) the equivalence classes for relation 1 (resp. 2).

# Stable Bernstein centre, after Haines

Let $\imath = [\widehat{M}, \lambda] \in \mathcal{B}(G)_{\mathrm{st}}$. We can define the torus

$$\mathcal{T}_\imath = \{(\lambda\chi)_{\widehat{M}},\ \chi \in \mathfrak{X}(\widehat{M})\}$$

and the finite group

$$\mathcal{W}_\imath = \left\{w \in N_{\widehat{G}}(\widehat{M})/\widehat{M}, \exists \chi \in \mathfrak{X}(\widehat{M}), ({}^w\lambda)_{\widehat{M}} = (\lambda\chi)_{\widehat{M}}\right\}.$$

We have

$$\Omega(G)_{\mathrm{st}} \simeq \bigsqcup_{\imath \in \mathcal{B}(G)_{\mathrm{st}}} \mathcal{T}_\imath/\mathcal{W}_\imath.$$

## Définition

We call the stable Bernstein centre of $G$ and we denote by $\mathfrak{Z}(G)_{\mathrm{st}}$ the ring of regulars functions on $\Omega(G)_{\mathrm{st}}$ :

$$\mathfrak{Z}(G)_{\mathrm{st}} := \mathbb{C}[\Omega(G)_{\mathrm{st}}].$$

# Stable Bernstein centre, after Haines

Let $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \widehat{G}$ a Langlands parameter.

$$\begin{array}{rccc} \lambda_\phi : & W_F & \longrightarrow & \widehat{G} \\ & w & \longmapsto & \phi\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right) \end{array}.$$

We denote by $\widehat{M}_{\lambda_\phi}$ a Levi subgroup of $\widehat{G}$ which contains minimally the image of $\lambda_\phi$.

$$\begin{array}{rccc} \mathscr{L}_{\mathrm{st}} : & \Phi(G) & \longrightarrow & \Omega(G)_{\mathrm{st}} \\ & \phi & \longmapsto & (\widehat{M}_{\lambda_\phi}, \lambda_\phi)_{\widehat{G}} \end{array},$$

$$\Phi(G) = \bigsqcup_{(\widehat{M},\lambda)\in\Omega(G)_{\mathrm{st}}} \Phi(G)_\lambda, \quad \text{where} \quad \Phi(G)_\lambda = \{\phi \in \Phi(G), \mathscr{L}_{\mathrm{st}}(\phi) = (\widehat{M}, \lambda)\}$$

$$\Phi(G) = \bigsqcup_{\imath\in\mathcal{B}(G)_{\mathrm{st}}} \Phi(G)_\imath, \quad \text{where} \quad \Phi(G)_\imath = \{\phi \in \Phi(G), \mathscr{L}_{\mathrm{st}}(\phi) \in \imath\}.$$

# Compatibility of the Langlands correspondence with the parabolic induction

## Compatibility conjecture

Let $P = LU$ be a parabolic subgroup of $G$, $\sigma \in \mathrm{Irr}(L)$ supercuspidal and $\pi$ an irreducible subquotient of $i_P^G(\sigma)$.

- $\phi_\sigma : W_F' \longrightarrow \widehat{L}$ Langlands parameter of $\sigma$ ;
- $\phi_\pi : W_F' \longrightarrow \widehat{G}$ Langlands parameter of$\pi$ ;

Then, $(\lambda_{\phi_\sigma})_{\widehat{G}} = (\lambda_{\phi_\pi})_{\widehat{G}}$.

Let $\imath = [\widehat{M}, \lambda] \in \mathcal{B}(G)_{\mathrm{st}}$.

$$\widetilde{\Pi}_\lambda(G) = \bigsqcup_{\lambda=\lambda_\phi} \Pi_\phi(G).$$

$$\widetilde{\Pi}_\imath(G) = \bigsqcup_{\lambda\chi\in\mathcal{T}_\imath/\mathcal{W}_\imath} \widetilde{\Pi}_{\lambda\chi}(G).$$

# Compatibility of the Langlands correspondence with the parabolic induction

$$\widetilde{\Pi}_\lambda(G) = \bigsqcup_{\lambda=\lambda_\phi} \Pi_\phi(G).$$

## Proposition

The compatibility conjecture is equivalent to that for all Levi subgroup $\widehat{M}$ of $\widehat{G}$ and all $\lambda : W_F \longrightarrow \widehat{M}$ discrete, we have :

$$\widetilde{\Pi}_\lambda(G) = \bigsqcup_{\widehat{L}\in\mathcal{L}(G)_\lambda} \bigsqcup_{\phi\in\Phi(L)_{\lambda,\mathrm{cusp}}} \bigsqcup_{\pi\in\Pi_\phi(L)_{\mathrm{cusp}}} \mathcal{JH}(i_{LU}^G(\pi))$$

# Enhanced Bernstein centre

We still suppose that $G$ is a split connected reductive $p$-adic group and let $L$ be a Levi subgroup of $G$.

## Définition (M.)

Let $\varphi \in \Phi(L)$. We say that $\varphi$ is cuspidal when

- $\varphi$ is discrete ;
- $\mathrm{Irr}(\mathcal{S}_\varphi^L)_{\mathrm{cusp}}$ is not empty.

An enhanced Langlands parameter is cuspidal $(\varphi, \varepsilon) \in \Phi(L)^+$ when $\varphi$ is cuspidal and $\varepsilon \in \mathrm{Irr}(\mathcal{S}_\varphi^L)_{\mathrm{cusp}}$.

## Conjecture (M.)

Let $\varphi \in \Phi(L)$. The $L$-packet $\Pi_\varphi(L)$ contains supercuspidal representations, if and only if, $\varphi$ is a cuspidal Langlands parameter. Moreover, the supercuspidal representations in $\Pi_\varphi(L)$ are parametrized by $\mathrm{Irr}(\mathcal{S}_\varphi^L)_{\mathrm{cusp}}$,
$\Phi_\varphi(G)_{\mathrm{cusp}} \simeq \mathrm{Irr}(\mathcal{S}_\varphi^L)_{\mathrm{cusp}}$.

# Enhanced Bernstein centre

## Proposition (M.)

Let $\lambda : W_F \longrightarrow \widehat{M}$ a discrete parameter.
If $\phi \in \Phi(M)$ is a Langlands parameter of $M$ with infinitesimal cocharacter $\lambda$, then $\phi = \lambda$.

$$\widetilde{\Pi}_\lambda(M) = \Pi_\lambda(M).$$

Moreover, all representations of $\mathcal{S}_\lambda(M)$ are cuspidal, i.e.

$$\mathrm{Irr}(\mathcal{S}_\lambda^M) = \mathrm{Irr}(\mathcal{S}_\lambda^M)_{\mathrm{cusp}}.$$

# Enhanced Bernstein centre

## Proposition (M.)

For the linear group, symplectic or special orthogonal split group, the cuspidal Langlands parameters are :

- for $\mathrm{GL}_n(F)$,

$$\varphi : W_F \longrightarrow \mathrm{GL}_n(\mathbb{C}), \text{ irréductible;}$$

- for $\mathrm{SO}_{2n+1}(F)$,

$$\varphi = \bigoplus_{\pi \in I_O} \bigoplus_{a=1}^{d_\pi} \pi \boxtimes S_{2a} \bigoplus_{\pi \in I_S} \bigoplus_{a=1}^{d_\pi} \pi \boxtimes S_{2a-1}, \ \forall \pi \in I_O, d_\pi \in \mathbb{N}, \ \forall \pi \in I_S, d_\pi \in \mathbb{N}^*;$$

- for $\mathrm{Sp}_{2n}(F)$ ou $\mathrm{SO}_{2n}(F)$,

$$\varphi = \bigoplus_{\pi \in I_S} \bigoplus_{a=1}^{d_\pi} \pi \boxtimes S_{2a} \bigoplus_{\pi \in I_O} \bigoplus_{a=1}^{d_\pi} \pi \boxtimes S_{2a-1} \ \forall \pi \in I_O, d_\pi \in \mathbb{N}^*, \ \forall \pi \in I_S, d_\pi \in \mathbb{N}.$$

## Proposition

Moreover, after the theorem of Harris-Taylor et Henniart for $\mathrm{GL}$ and the theorem of Mœglin, the supercuspidal representations of $G$ are parametrized by $(\varphi, \varepsilon)$ with $\varphi$ a cuspidal Langlands parameter of $G$ and $\varepsilon \in \mathrm{Irr}(\mathcal{S}_{\varphi}^{G})_{\mathrm{cusp}}$. In other words, the conjecture on the parametrization of supercuspidal representations is true.

# Enhanced Bernstein centre

Let $G$ be a split reductive $p$-adic group.
$(\widehat{L}, \varphi, \varepsilon)$ with $\widehat{L}$ a Levi subgroup of $\widehat{G}$ and $(\varphi, \varepsilon) \in \Phi(L)^+$ cuspidal.
Unramified cocharacters $\mathfrak{X}(\widehat{L}) = \{\chi : W_F/I_F \longrightarrow Z^\circ_{\widehat{L}}\} \simeq Z^\circ_{\widehat{L}}$.

## Définition

1. the cuspidal $L$-data $(\widehat{L}_1, \varphi_1, \varepsilon_1)$ and $(\widehat{L}_2, \varphi_2, \varepsilon_2)$ are associate if there exists $g \in \widehat{G}$ such that ${}^g\widehat{L}_1 = \widehat{L}_2$, ${}^g\varphi_1 = \varphi_2$ et $\varepsilon_1^g \simeq \varepsilon_2$ ;
2. the cuspidal $L$-data $(\widehat{L}_1, \varphi_1, \varepsilon_1)$ and $(\widehat{L}_2, \varphi_2, \varepsilon_2)$ are inertially equivalente if there existsg $g \in \widehat{G}$ et $\chi \in \mathfrak{X}(\widehat{L}_2)$ such that ${}^g\widehat{L}_1 = \widehat{L}_2$, ${}^g\varphi_1 = \varphi_2\chi$ et $\varepsilon_1^g \simeq \varepsilon_2$ ;

We denote by $\Omega(G)^+_{\mathrm{st}}$ (resp. $\mathcal{B}(G)^+_{\mathrm{st}}$) the equivalences classes for relation 1 (resp. 2).

# Enhanced Bernstein centre

Let $\jmath = [\widehat{L}, \varphi, \varepsilon] \in \mathcal{B}(G)^+_{\mathrm{st}}$. We can define the torus

$$\mathcal{T}_\jmath = \{(\varphi\chi)_{\widehat{L}},\ \chi \in \mathfrak{X}(\widehat{L})\}$$

and the finite group

$$\mathcal{W}_\jmath = \left\{ w \in N_{\widehat{G}}(\widehat{L})/\widehat{L}, \exists \chi \in \mathfrak{X}(\widehat{L}), ({}^w\varphi)_{\widehat{L}} = (\varphi\chi)_{\widehat{L}} \right\}.$$

We have

$$\Omega(G)^+_{\mathrm{st}} \simeq \bigsqcup_{\jmath \in \mathcal{B}(G)^+_{\mathrm{st}}} \mathcal{T}_\jmath / \mathcal{W}_\jmath.$$

# Enhanced Bernstein centre

We have the following map

$$\begin{array}{ccc} \Phi(G) & \longrightarrow & \Omega(G)_{\mathrm{st}} \\ \phi & \longmapsto & (\widehat{M}_{\lambda_\phi}, \lambda_\phi)_{\widehat{G}} \end{array},$$

$$\begin{array}{ccc} \Omega(G)^+_{\mathrm{st}} & \longrightarrow & \Omega(G)_{\mathrm{st}} \\ (\widehat{L}, \varphi, \varepsilon)_{\widehat{G}} & \longmapsto & (\widehat{M}_{\lambda_\varphi}, \lambda_\varphi)_{\widehat{G}} \end{array},$$

$$\begin{array}{ccc} \Phi(G)^+ & \longrightarrow & \Omega(G)^+_{\mathrm{st}} \\ (\phi, \eta) & \longmapsto & (\widehat{L}, \varphi, \varepsilon)_{\widehat{G}} \end{array}, \quad ?????$$

# Enhanced Bernstein centre

## Conjecture

Let $\varphi : W'_F \longrightarrow \widehat{L}$ be a cuspidal Langlands parameter of $L$. Assume the conjecture (on the parametrization of supercuspidal) true. If $\sigma \in \Pi_\varphi(L)_{\mathrm{cusp}}$ is parametrized by $\varepsilon \in \mathrm{Irr}(\mathcal{S}_\varphi^L)_{\mathrm{cusp}}$, then if we denote by $\mathfrak{s} = [L, \sigma]_G$, $\jmath = [\widehat{L}, \varphi, \varepsilon]$, we have isomorphism :

$$\begin{array}{ccc} T_\mathfrak{s} & \longrightarrow & \mathcal{T}_\jmath \\ \chi & \longmapsto & \widehat{\chi} \end{array}, \qquad \begin{array}{ccc} W_\mathfrak{s} & \longrightarrow & \mathcal{W}_\jmath \\ w & \longmapsto & \widehat{w} \end{array},$$

such that for all $\chi \in T_\mathfrak{s}$, $w \in W_\mathfrak{s}$ :

$$\widehat{w \cdot \chi} = \widehat{w} \cdot \widehat{\chi}.$$

$$\Omega(G) \simeq \Omega(G)^+_{\mathrm{st}}$$

# Cuspidal support of an enhanced Langlands parameter

## Theorem (M.)

Let $G$ be a split reductive $p$-adic group. We can define a map

$$(\phi, \eta) \longmapsto (\widehat{L}, \varphi, \varepsilon_0)_{\widehat{G}},$$

with

- $\widehat{L}$ a Levi subgroup of $\widehat{G}$ ;
- $\varphi \in \Phi(L)$a cuspidal Langlands parameter of $L$ ;
- an irreducible representation $\varepsilon_0$ of $A_{Z_{\widehat{L}}(\varphi_{|W_F})^\circ}(\varphi_{|\mathrm{SL}_2(\mathbb{C})})$.

The Langlands parameters $\phi$ and $\varphi$ have the same infinitesimal cocharacter and for all $w \in W_F$, we have

$$\chi_c(w) = \phi(1, d_w)/\varphi(1, d_w) \in Z^\circ_{\widehat{L}},$$

with $d_w = \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}$

# Cuspidal support of an enhanced Langlands parameter

## Theorem (M.)

Let $G$ be one of $\mathrm{Sp}_{2n}(F)$ ou $\mathrm{SO}_n(F)$. We can define a map

$$\begin{array}{rccc} \mathscr{S}_c : & \Phi(G)^+ & \longrightarrow & \Omega(G)^+_{\mathrm{st}} \\ & (\phi, \eta) & \longmapsto & (\widehat{L}, \varphi, \varepsilon) \end{array}$$

Moreover, the fiber are parametrized by irreducible representations of

$$N_{Z_{\widehat{G}}(\varphi_{|W_F}\chi_c)}(A_{\widehat{L}})/Z_{\widehat{L}}(\varphi_{|W_F}\chi_c),$$

where $c$ runs over the correcting cocharacter of $\varphi$ in $\widehat{G}$.

# Equivalence of categories

Let $G$ be $\mathrm{Sp}_{2n}(F)$ or $\mathrm{SO}_n(F)$, $M = \mathrm{GL}_{d_1}^{\ell_1} \times \ldots \times \mathrm{GL}_{d_r}^{\ell_r} \times G_{n'}$ Levi subgroup of $G$ and

$$\sigma = \underbrace{\sigma_1 \boxtimes \ldots \boxtimes \sigma_1}_{\ell_1} \boxtimes \ldots \boxtimes \underbrace{\sigma_r \boxtimes \ldots \boxtimes \sigma_r}_{\ell_r} \boxtimes \tau,$$

with $\sigma_i$ unitary irreducible supercuspidal representation of $\mathrm{GL}_{d_i}$ and $\tau$ supercuspidal irreducible representation of $G_{n'}$.
We denote by $\mathfrak{s} = [M, \sigma]_G$. Heiermann associate to each $\mathfrak{s}$ :

- a based root datum $\mathcal{R}_{\mathfrak{s}} = (X_{\mathfrak{s}}, \Sigma_{\mathfrak{s}}, X^{\vee}, \Sigma_{\mathfrak{s}}^{\vee}, \Delta_{\mathfrak{s}})$ ;
- a finite group $R_{\mathfrak{s}}$ ;
- parameters functions $(\lambda_{\mathfrak{s}}, \lambda_{\mathfrak{s}}^*)$
- an affine Hecke algebra $\mathcal{H}_{\mathfrak{s}}$.

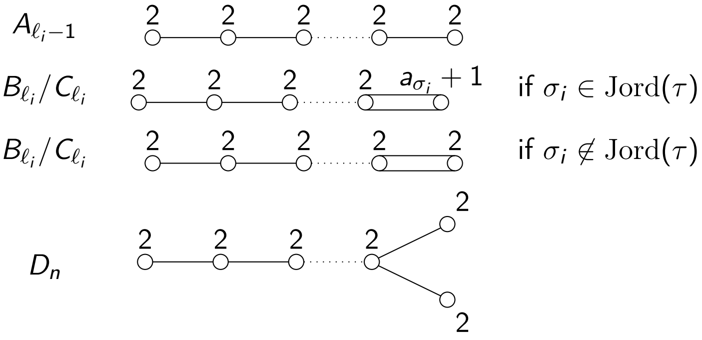
# Equivalence of categories

## Theorem (Heiermann)

The category $\mathrm{Rep}(G)_{\mathfrak{s}}$ is equivalent to the category of right $\mathcal{H}_{\mathfrak{s}} \rtimes \mathbb{C}[R_{\mathfrak{s}}]$-modules.
This equivalence preserve the objects of the discrete serie and tempered objects.

- We have $\Sigma_{\mathfrak{s}} = \bigsqcup_{i=1}^{r} \Sigma_i$ ;
- If $\Sigma_i$ is type $A, C, D$ or (type $B$ for long roots), $\alpha \in \Sigma_i \cap \Delta_{\mathfrak{s}}$ $\lambda_{\mathfrak{s}}(\alpha) = 1$ ;
- If $\Sigma_i$ is type $B$, for the short root $\lambda_{\mathfrak{s}}(\alpha_i) = x_i + x_i',\ \ \lambda^*_{\mathfrak{s}}(\alpha_i) = x_i - x_i'$, with $x_i$ the unique positive real number $x$ such that $\sigma_i|\ |^x \rtimes \tau$ reducible (same for $x_i'$ with $\sigma_i\zeta$).

# Parameter of the graded Hecke algebra obtained by $\mathcal{H}_{\mathfrak{s}}$




$$a_{\sigma_i} = \sup_{a \in \mathbb{N}} \{(\pi, a) \in \mathrm{Jord}(\tau)\}$$

# Graded Hecke algebra associated to cuspidal triple

Let $H$ be a connected reductive complex group and $\mathfrak{t} = (L, \mathcal{C}, \varepsilon) \in \mathcal{S}_H$.
Let $\mathfrak{h}$ the Lie algebra of $H$ et $(\sigma, r_0) \in \mathfrak{h} \oplus \mathbb{C}$ a semi-simple element.

$$\{x \in \mathfrak{h},\ [\sigma, x] = 2r_0 x\}.$$
$$(x, \eta),\ \eta \in \mathrm{Irr}(A_H(\sigma, x)).$$

From $\mathfrak{t} = (L, \mathcal{C}, \varepsilon)$ Lusztig build a :

- based root datum $\mathcal{R} = (X, \Sigma, X^\vee, \Sigma^\vee, \Delta)$;
- a parameter function $\mu_\mathfrak{t} : \Delta \longrightarrow \mathbb{N}$;
- a graded Hecke algebra $\mathbb{H}_{\mu_\mathfrak{t}}$.

Let $(\sigma, r_0) \in \mathfrak{h} \oplus \mathbb{C}$ a semisimple element.
Lusztig defined a $\mathbb{H}_{\mu_\mathfrak{t}}$-module $\mathbb{M}(\sigma, r_0, x)$. Let $\eta \in \mathrm{Irr}(A_H(\sigma, x))$ and

$$\mathbb{M}(\sigma, r_0, x, \eta) = \mathrm{Hom}_{A_H(\sigma,x)}(\eta, \mathbb{M}(\sigma, r_0, x)).$$

Let $\mathrm{Irr}(A_H(x))_\varepsilon$ the irreducible representation $\widetilde{\eta}$ of $A_H(x)$ such that $(\mathcal{C}_x^H, \widetilde{\eta})$ is in the bloc defined by $(L, \mathcal{C}, \varepsilon)$.
We have $A_H(\sigma, x) \hookrightarrow A_H(x)$ and we denote by $\mathrm{Irr}(A_H(\sigma, r_0, x))_\varepsilon$ the set of irreducible representation of $A_H(\sigma, r_0, x)$ which appears in the restriction to

# Graded Hecke algebra associated to cuspidal triple

## Theorem (Lusztig)

1. $\mathbb{M}(\sigma, r_0, x, \eta) \neq 0$, iff, $\eta \in \mathrm{Irr}(A_H(\sigma, r_0, x))_\varepsilon$
2. All simple $\mathbb{H}_{\mu_\mathfrak{t}}$-module on which $r$ acts by $r_0$ is a quotient of $\overline{\mathbb{M}}(\sigma, r_0, x, \eta)$ of one $\mathbb{M}(\sigma, r_0, x, \eta)$, with $\eta \in \mathrm{Irr}(A_H(\sigma, r_0, x))_\varepsilon$
3. The set of simple $\mathbb{H}_{\mu_\mathfrak{t}}$-modules with central character $(\sigma, r_0)$ is in bijection with
$$\mathcal{M}_{(\sigma,r_0)} = \{(x, \eta) | x \in \mathfrak{h}, [\sigma, x] = 2r_0 x, \eta \in \mathrm{Irr}(A_H(\sigma, x))_\varepsilon\}$$
4. A simple $\mathbb{H}_{\mu_\mathfrak{t}}$-module $\overline{\mathbb{M}}(\sigma, r_0, x, \eta)$ est tempered, iff, there exists a $\mathfrak{sl}_2$-triple $(x, h, y)$ in $\mathfrak{h}$ such that $[\sigma, x] = 2r_0 x,\ [\sigma, h] = 0,\ [\sigma, y] = -2r_0 y$ and $\sigma - r_0 h$ is elliptic. In this case, $\overline{\mathbb{M}}(\sigma, r_0, x, \eta) = \mathbb{M}(\sigma, r_0, x, \eta)$
5. If $\mathcal{C}_x^H$ is a distinguished nilpotent orbit of $H$, then $\mathbb{M}(\sigma, r_0, x, \eta)$ is in discrete serie.

# Graded Hecke algebra associated to cuspidal triple

| $H$ | $L$ | partition | $R$ | $R_{\mathrm{red}}$ | paramètres |
|---|---|---|---|---|---|
| $\mathrm{Sp}_{2n}$ | $(\mathbb{C}^\times)^\ell \times \mathrm{Sp}_{2n'}$ | $(1^\ell) \times (2, 4, \ldots, 2d)$ | $BC_\ell$ | $B_\ell$ | $2 - 2 - 2 \cdots 2 \Rightarrow 2d+1$ |
| | $(\mathbb{C}^\times)^n$ | $(1^n)$ | $C_n$ | $C_n$ | $2 - 2 - 2 \cdots 2 \Rightarrow 2$ |
| $\mathrm{SO}_N$ | $(\mathbb{C}^\times)^\ell \times \mathrm{SO}_{N'}$ | $(1^\ell) \times (1, 3, \ldots, 2d+1)$ | $B_\ell$ | $B_\ell$ | $2 - 2 - 2 \cdots 2 \Rightarrow 2d+2$ |
| $\mathrm{SO}_{2n+1}$ | $(\mathbb{C}^\times)^n$ | $(1^n)$ | $B_n$ | $B_n$ | $2 - 2 - 2 \cdots 2 \Rightarrow 2$ |
| $\mathrm{SO}_{2n}$ | $(\mathbb{C}^\times)^n$ | $(1^n)$ | $D_n$ | $D_n$ | $2 - 2 - 2 \cdots 2 <$ (2, 2) |

## Theorem (M.)

Let be $G$ a split classical group. Let $\mathfrak{s} = [L, \sigma] \in \mathcal{B}(G)$ and the corresponding $\jmath = [\widehat{L}, \varphi, \varepsilon] \in \mathcal{B}_e^{\mathrm{st}}(G)$. We have a bijection

$$\mathrm{Irr}(G)_{\mathfrak{s}} \simeq \Phi_e(G)_{\jmath},$$

which induced a bijection

$$\mathrm{Irr}(G)_{\mathfrak{s},2} \simeq \Phi_e(G)_{\jmath,2},$$

and

$$\mathrm{Irr}(G)_{\mathfrak{s},\mathrm{temp}} \simeq \Phi_e(G)_{\jmath,\mathrm{bdd}}.$$

## Theorem (M.)

The compatibility conjecture between the Langlands correspondence and the parabolic induction is true for the split classical groups.

Thank your for your attention.